Outer-product-of-gradients tests for spatial autoregressive models

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Abstract

For Lagrangian multiplier (LM) tests of restrictions on parameters in spatial autoregressive (SAR) models with (SARAR models) or without SAR disturbances, their outer-product-of-gradient (OPG) variants can be simple and robust to unknown heteroskedasticity. However, for certain tests, asymptotic distributions of test statistics might depend on the constrained maximum likelihood or quasi maximum likelihood (QML) estimators, so their OPG variants would not be valid. To overcome such a hurdle, we propose to use $C(\alpha)$-type score vectors to obtain valid OPG variants. Such OPG tests can be systematically constructed for SARAR models with homoskedastic and heteroskedastic disturbances, which might not be normally distributed. They also have the advantage that any $\sqrt{n}$-consistent estimator can be used in place of a restricted QML estimate. In particular, OPG tests based on generalized method of moments (GMM) estimates are computationally simple and powerful compared to LM tests. Corresponding OPG tests based on $C(\alpha)$-type gradient vectors in the GMM framework are also investigated.

Keywords: LM test, OPG, $C(\alpha)$ test, GMM test, unknown heteroskedasticity, spatial dependence

JEL classification: C12, C13, C14, C21, C52

1 Introduction

This paper proposes to use the $C(\alpha)$-type statistic formulation to derive outer-product-of-gradients (OPG) tests for spatial autoregressive (SAR) models. With the formulation, all kinds of valid tests for SAR models can be derived. The tests are simple and can be robust to unknown heteroskedasticity. In addition, any $\sqrt{n}$-consistent estimator of nuisance parameters can be used. Thus, computationally simpler estimators such as the generalized method of moments (GMM) estimator can be used to construct test statistics.

OPG tests for spatial econometric models considered in the literature are limited to certain tests. Born and Breitung (2011) propose to use OPG variance estimates in forming Lagrangian multiplier (LM) test statistics for

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spatial dependence. They consider LM tests for no spatial dependence in the SAR and spatial error (SE) models, and no spatial lag and spatial error dependence at the same time in the SAR model with SAR disturbances (SARAR model). For these tests, a test statistic can be decomposed into a sum of asymptotically uncorrelated terms, and a new test statistic can be formed by using the sum of squares of these terms as a variance estimate of the test statistic. Thus the test statistic can be computed as a product of a simple regression. The OPG variants relate to the martingale structure of linear-quadratic forms, which is explored in Kelejian and Prucha (2001) to develop a central limit theorem for spatial econometric statistics.[1] Baltagi and Yang (2013) introduce general methods to modify standard LM tests so that they are robust to heteroskedasticity and non-normality and obtain their OPG variants. Neither Born and Breitung (2011) nor Baltagi and Yang (2013) have considered testing for only no spatial error dependence in the presence of spatial lag dependence or vice versa in the SARAR model. For these tests, we observe that an orthogonality condition is absent, thus the estimator for the nuisance parameter vector involved in the testing score will have an impact on its asymptotic distribution. Then simple OPG variants of LM tests, formed by a sum of squares of components for an LM test statistic, may not be valid (Baltagi and Yang, 2013, p. 734 in its conclusion).

However, $C(\alpha)$-type statistic formulation based on the quasi maximum likelihood (QML) score vector can be used to overcome the problem (Neyman, 1959). Even though the C(α)-type statistic evaluated at the restricted QML estimate will be identically the LM statistic, a C(α)-type formulation modifies the LM statistic formulation at the true parameter vector such that additional terms can take care of the asymptotic influence of the QML estimate on the statistic. With the martingale structure of the spatial econometric statistic, the variance of the test can then be consistently estimated with OPG and an asymptotically chi-squared distributed statistic can be constructed. This OPG test has the same asymptotic distribution as the LM test. It does not require an analytic formula of the variance and is easy to compute. For this score-based $C(\alpha)$-type test statistic, any $\sqrt{n}$-consistent estimator for the nuisance parameter vector will not have impact on the asymptotic distribution of the test statistic either. Thus, for SARAR models we can also use computationally simple estimators, such as the spatial two stage least squares (2SLS) estimator (Kelejian and Prucha, 1998; Lee, 2007) or GMM estimator (Lee, 2007), instead of the maximum likelihood (ML) or QML estimator (Lee, 2004a), to construct test statistics.

A score-based OPG test constructed under homoskedastic variances of an SARAR model is robust to non-normality, but might not be robust to unknown heteroskedasticity if its expected score at the true parameter vector were not zero. This case can occur because the model is misspecified and the corresponding likelihood function might not even be a quasi-likelihood. A possible test statistic may then be based on some proper moment conditions (Kelejian and Prucha, 2010; Lin and Lee, 2010). An alternative is to follow Baltagi and Yang (2013) by modifying the score vectors so that their expected values are zero at the true parameter vector, and consider robust $C(\alpha)$-type tests based on those modified score vectors. The modified score statistics are effectively moment test statistics. In the case that the asymptotic distribution of those estimated moments depends on an initial estimator, we suggest

[1] Quadratic statistics have long been written as martingales in the time series literature (see, e.g., Hall and Heyde, 1980). We thank an anonymous referee for pointing this out.

the formulation of a $C(\alpha)$-type moment vector such that the resulted empirical moment vector is asymptotically invariant with respect to any $\sqrt{n}$-consistent estimator.[2] The OPG variant will be valid as long as the $C(\alpha)$-type moment vector at the true parameter vector can be expressed as a sum of martingale difference arrays.

We also consider OPG tests based on $C(\alpha)$-type gradients in the GMM framework under both homoskedasticity and unknown heteroskedasticity. The GMM criterion functions involve moment vectors that are linear and quadratic in disturbances, so OPG tests are derived with a similar principle.

In practice, researchers might use other tests such as the Wald, LM and likelihood ratio (LR) tests in the likelihood framework. The three classical tests are asymptotically equivalent under both the null and local alternatives, and are locally most powerful.[3] However, in finite samples, they may differ in size and power properties. For computation, the Wald test needs to estimate the unrestricted model, the LM test only needs to estimate the restricted model under the null hypothesis, which is often simpler, and the LR test needs to estimate both. For SARAR models, these estimations all involve evaluations of log determinants, which can be computationally intensive for large sample sizes. In the GMM framework, the corresponding tests are the Wald, gradient and distance difference tests. Similar comments apply in terms of whether the unrestricted and/or restricted models are estimated. We conduct a comprehensive Monte Carlo study to investigate the finite sample performance of the proposed OPG tests and compare them with other tests. Several different consistent estimators are used for OPG tests. In particular, we have considered GMM estimators generated from moment vectors that are the best ones for homoskedastic normal disturbances. Score-based OPG tests with these GMM estimators perform well and are competitive alternatives to other classical tests in finite samples.

This paper is organized as follows. Section 2 proposes to use the $C(\alpha)$-type score vectors to systematically derive OPG tests in the SARAR model under both homoskedasticity and unknown heteroskedasticity. We derive test statistics for detecting spatial error and spatial lag dependence in the SARAR model. Section 3 considers gradient-based OPG tests in the GMM framework. Section 4 reports Monte Carlo results. Section 5 concludes. Lemmas and proofs are collected in the appendices.

2 Score-based OPG tests

In this section, we first consider the SARAR model with homoskedastic disturbances. We illustrate that $C(\alpha)$-type score vectors can be used to systematically derive OPG variants of test statistics. We next consider OPG tests under unknown heteroskedasticity. Our main interest is to derive OPG tests for detecting spatial error and spatial lag dependence.

The SARAR model is as follows:

$$Y_n = \lambda W_n Y_n + X_n \beta + U_n, \quad U_n = \rho M_n U_n + V_n, \tag{1}$$

[2]In the spatial econometrics literature, Lee and Yu (2012) have considered $C(\alpha)$-type gradient tests in the GMM framework for spatial lag dependence in the SAR model.

[3]See, e.g., van der Vaart (1998, Chapters 15–16) and Ruud (2000, Chapter 17).

where n is the sample size, Y_n is an $n \times 1$ vector of observations on the dependent variable, X_n is an $n \times k$ matrix of exogenous variables with corresponding parameter vector β, $W_n = (w_{n,ij})$ and $M_n = (m_{n,ij})$ are $n \times n$ nonstochastic spatial weights matrices with zero diagonals, λ and ρ are scalar spatial dependence parameters, and $V_n = (v_{ni})$ is an $n \times 1$ vector of independent disturbances with mean zero.[4] Let $\delta = (\lambda, \rho, \beta')'$, δ_0 be the true value of δ, $S_n(\lambda) = I_n - \lambda W_n$, and $R_n(\rho) = I_n - \rho M_n$, where I_n is the $n \times n$ identity matrix. Model (1) is an equilibrium model. Thus, Y_n has the reduced form $Y_n = S_n^{-1}(X_n\beta_0 + R_n^{-1}V_n)$, where $S_n = S_n(\lambda_0)$ and $R_n = R_n(\rho_0)$ are assumed to be invertible. The X_n is assumed to be nonstochastic for convenience, as in Kelejian and Prucha (1998) and Lee (2004a).[5] The SAR model is (1) with $\rho = 0$, and the SE model is (1) with $\lambda = 0$.

The log QML function as if the disturbances were i.i.d. normal with variance σ^2 is

$$\ln L_n(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \ln|S_n(\lambda)| + \ln|R_n(\rho)| - \frac{1}{2\sigma^2}[S_n(\lambda)Y_n - X_n\beta]' R_n'(\rho) R_n(\rho)[S_n(\lambda)Y_n - X_n\beta], \quad (2)$$

where $\theta = (\delta', \sigma^2)'$. Under the assumption that $V_n \sim (0, \sigma_0^2 I_n)$, the first order derivatives of $\ln L_n(\theta)$ at $\theta_0 = (\delta_0', \sigma_0^2)'$ are

$$\frac{\partial \ln L_n(\theta_0)}{\partial \lambda} = \frac{1}{\sigma_0^2} V_n' \ddot{G}_n V_n - \operatorname{tr}(G_n) + \frac{1}{\sigma_0^2}(R_n G_n X_n \beta_0)' V_n, \quad (3)$$

$$\frac{\partial \ln L_n(\theta_0)}{\partial \rho} = \frac{1}{\sigma_0^2} V_n' H_n V_n - \operatorname{tr}(H_n), \quad (4)$$

$$\frac{\partial \ln L_n(\theta_0)}{\partial \beta} = \frac{1}{\sigma_0^2} X_n' R_n' V_n, \quad (5)$$

$$\frac{\partial \ln L_n(\theta_0)}{\partial \sigma^2} = \frac{1}{2\sigma_0^4} V_n' V_n - \frac{n}{2\sigma_0^2}, \quad (6)$$

where $G_n = W_n S_n^{-1}$, $\ddot{G}_n = R_n G_n R_n^{-1}$ and $H_n = M_n R_n^{-1}$. The LM and OPG tests can be based on these scores. If the disturbances are heteroskedastic, we need to modify these scores to obtain valid test statistics.

2.1 Tests under homoskedasticity

In this section, we consider test statistics for the SARAR model with homoskedastic disturbances. We first show that $C(\alpha)$-type score vectors can be used conveniently to derive OPG tests and then derive test statistics for $\rho_0 = 0$ and/or $\lambda_0 = 0$.

Note that each element of $\frac{\partial \ln L_n(\theta_0)}{\partial \theta}$ is a linear-quadratic form $V_n' A_n V_n - \sigma_0^2 \operatorname{tr}(A_n) + b_n' V_n$, where A_n is an $n \times n$ nonstochastic matrix, and b_n is an $n \times 1$ nonstochastic vector. Thus, consider a vector of linear-quadratic forms

$$\Xi_n = [V_n' A_{n1} V_n - \sigma_0^2 \operatorname{tr}(A_{n1}) + b_{n1}' V_n, \ldots, V_n' A_{np} V_n - \sigma_0^2 \operatorname{tr}(A_{np}) + b_{np}' V_n]'$$

for some finite p, where $A_{nr} = (a_{nr,ij})$ and $b_{nr} = (b_{nr,i})$ for $r = 1, \ldots, p$. We can rewrite Ξ_n as a sum of martingale differences. Specifically $\Xi_n = \sum_{i=1}^n \xi_{ni}$, where

$$\xi_{ni} = [a_{n1,ii}(v_{ni}^2 - \sigma_0^2) + v_{ni}\sum_{j=1}^{i-1}(a_{n1,ij} + a_{n1,ji})v_{nj} + b_{n1,i}v_{ni}, \ldots, a_{np,ii}(v_{ni}^2 - \sigma_0^2) + v_{ni}\sum_{j=1}^{i-1}(a_{np,ij} + a_{np,ji})v_{nj} + b_{np,i}v_{ni}]'.$$

[4] To allow for possible unknown heteroskedastic variances, the disturbances are assumed to be independent but not necessarily identically distributed.

[5] Alternatively, X_n can be assumed to be stochastic with finite moments of certain order.

Consider the σ-fields $\mathscr{F}_{n0} = \{\emptyset, \Omega\}$, $\mathscr{F}_{ni} = \sigma(v_{n1}, \ldots, v_{ni})$, $1 \le i \le n$. As $\mathscr{F}_{n,i-1} \subset \mathscr{F}_{ni}$ and $\mathrm{E}(\xi_{ni}|\mathscr{F}_{n,i-1}) = 0$, $\{\xi_{ni}, \mathscr{F}_{ni}, 1 \le i \le n, n \ge 1\}$ forms a martingale difference array. Thus, ξ_{ni}'s are uncorrelated and the variance of Ξ_n is

$$\mathrm{var}(\Xi_n) = \sum_{i=1}^{n} \mathrm{E}(\xi_{ni}\xi'_{ni}) = \mathrm{E}(\varphi'_n \varphi_n),$$

where $\varphi_n = (\xi_{n1}, \ldots, \xi_{nn})'$ is a matrix consisting of martingale differences. For any $n \times n$ matrix A_n, denote $A_n^{(s)} = A_n + A'_n$, $\mathrm{tril}(A_n)$ represents the strictly lower triangular matrix formed by the elements below the main diagonal of A_n, and $\mathrm{vec}_{\mathrm{D}}(A_n)$ denotes a column vector of the diagonal elements of A_n. Then φ_n can be conveniently written as

$$\begin{aligned}\varphi_n = [&V_n \circ \mathrm{vec}_{\mathrm{D}}(A_{n1}) \circ V_n - \sigma_0^2 \,\mathrm{vec}_{\mathrm{D}}(A_{n1}) + V_n \circ (\mathrm{tril}(A_{n1}^{(s)})V_n) + b_{n1} \circ V_n, \ldots, \\ &V_n \circ \mathrm{vec}_{\mathrm{D}}(A_{np}) \circ V_n - \sigma_0^2 \,\mathrm{vec}_{\mathrm{D}}(A_{np}) + V_n \circ (\mathrm{tril}(A_{np}^{(s)})V_n) + b_{np} \circ V_n],\end{aligned} \tag{7}$$

where $\circ$ denotes the Hadamard product. Denote $V_n(\delta) = R_n(\rho)[S_n(\lambda)Y_n - X_n\beta]$. For a consistent estimator $\hat{\theta}_n$ of θ_0, let $\hat{\varphi}_n$ be the matrix obtained from φ_n by replacing V_n with $V_n(\hat{\delta}_n)$, σ_0^2 with $\hat{\sigma}_n^2$, and θ_0 in A_{ni}'s and b_{ni}'s by $\hat{\theta}_n$.[6] Under regularity conditions, $\mathrm{var}(\Xi_n)$ can be estimated by the outer product

$$\hat{\varphi}'_n \hat{\varphi}_n = \sum_{i=1}^{n} \hat{\xi}_{ni}\hat{\xi}'_{ni}, \tag{8}$$

where $\hat{\xi}_{ni}$'s are derived similarly to $\hat{\varphi}_n$, such that $\frac{1}{n}\hat{\varphi}'_n\hat{\varphi}_n = \frac{1}{n}\mathrm{var}(\Xi_n) + o_p(1)$. The advantage of this variance estimator is that no analytical form of the variance is needed.[7] Since estimators and tests might relate to the scores in (3)–(6), relevant variances can be estimated with this OPG form.

Suppose that we are interested in testing whether the true value θ_{10} of a $k_1 \times 1$ subvector θ_1 of $\theta = (\theta'_1, \theta'_2)'$ is zero or not in the SARAR model (1). The LM test will be based on the asymptotic distribution of $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1}$, where $\tilde{\theta}_{2n}$ is the restricted QML estimator with the restriction $\theta_1 = 0$ imposed. By the mean value theorem, under regularity conditions,

$$\begin{aligned}\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1} &= \frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta_1} + \frac{1}{n}\frac{\partial^2 \ln L_n(0,\check{\theta}_{2n})}{\partial \theta_1 \partial \theta'_2}\sqrt{n}(\tilde{\theta}_{2n} - \theta_{20}) \\ &= \frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta_1} + \frac{1}{n}\mathrm{E}\Big(\frac{\partial^2 \ln L_n(0,\theta_{20})}{\partial \theta_1 \partial \theta'_2}\Big)\sqrt{n}(\tilde{\theta}_{2n} - \theta_{20}) + o_p(1),\end{aligned} \tag{9}$$

where $\check{\theta}_{2n}$ lies between $\tilde{\theta}_{2n}$ and θ_{20} elementwise. If $\frac{1}{n}\mathrm{E}(\frac{\partial^2 \ln L_n(0,\theta_{20})}{\partial \theta_1 \partial \theta'_2})$ does not go to zero, the asymptotic distribution of $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1}$ will generally be different from that of $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta_1}$. Because $\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta_1}$ is a vector of linear-quadratic forms as Ξ_n above such that $\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta_1} = \sum_{i=1}^n \xi_{n1,i}$ is a sum of martingale differences $\xi_{n1,i}$'s, its variance can be estimated as $\hat{\varphi}'_{n1}\hat{\varphi}_{n1}$, where $\hat{\varphi}_{n1} = (\hat{\xi}_{n1,1}, \ldots, \hat{\xi}_{n1,n})'$ and $\hat{\xi}_{n1,i}$'s are derived as above using the restricted estimator $(0, \tilde{\theta}'_{2n})'$. But $\frac{1}{n}\hat{\varphi}'_{n1}\hat{\varphi}_{n1}$ is generally not a consistent estimator for the limiting variance of $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1}$ as seen in (9) because the variance of the second term was ignored. Then the statistic

$$\Big(\sum_{i=1}^{n} \hat{\xi}_{n1,i}\Big)'\Big(\sum_{i=1}^{n} \hat{\xi}_{n1,i}\hat{\xi}'_{n1,i}\Big)^{-1}\Big(\sum_{i=1}^{n} \hat{\xi}_{n1,i}\Big) \tag{10}$$

[6] For the score vector $\frac{\partial \ln L_n(\theta_0)}{\partial \theta}$, from (3)–(6), vectors and square matrices in the linear-quadratic forms involve θ_0.

[7] The variance of a vector of linear quadratic forms such as $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(\theta_0)}{\partial \theta}$ in Kelejian and Prucha (2001) and Lee (2004a) are estimated with analytical forms. For non-normal disturbances, third and fourth moment parameters of v_{ni} will appear.

will not follow an asymptotic chi-squared distribution unless the orthogonality condition $\frac{1}{n}\,\mathrm{E}(\frac{\partial^2 \ln L_n(0,\theta_{20})}{\partial\theta_1 \partial\theta_2'}) = o(1)$ holds. If the orthogonality condition was satisfied, a test statistic with the form of (10) would have an asymptotic chi-squared distribution. In the orthogonality situation, $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial\theta_1}$ is written as a sum of approximate martingale differences and its variance can be estimated by a sum of outer products of these differences. For the tests in Born and Breitung (2011), either the orthogonality condition holds for the original statistic $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial\theta_1}$ or they can easily rewrite $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial\theta_1}$ in an equivalent form so that the impact of $\tilde{\theta}_{2n}$ on the asymptotic distribution is properly taken into account.[8] For other tests such as for $\lambda_0 = 0$ or $\rho_0 = 0$ in the SARAR model, the testing score generally cannot be conveniently transformed to obtain an OPG variant of the LM test.[9]

For the general testing problem of $H_0 : \theta_{10} = 0$, we suggest using the $C(\alpha)$-type score vector formulation so that the OPG variant of the LM test can be derived. The idea of the $C(\alpha)$-type test is to consider a test statistic modified from the LM statistic such that the orthogonality condition holds (Neyman, 1959). Let $\Omega_{n,12} = -\,\mathrm{E}(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\theta_1\partial\theta_2'})$ and $\Omega_{n,22} = -\,\mathrm{E}(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\theta_2\partial\theta_2'})$.[10] By imposing the restriction $\theta_{10} = 0$ but replacing θ_{20} in $\Omega_{n,12}$ and $\Omega_{n,22}$ by a $\sqrt{n}$-consistent estimator $\hat{\theta}_{2n}$, we obtain their estimates $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$. In particular, we can consider the $C(\alpha)$-type statistic

$$\hat{\Upsilon}_n = [I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\frac{\partial \ln L_n(0,\hat{\theta}_{2n})}{\partial\theta}. \tag{11}$$

By an expansion as in (9), under regularity conditions, $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n$ has the same asymptotic distribution as $\frac{1}{\sqrt{n}}\Upsilon_n = [I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\theta_{20})}{\partial\theta}$, so the asymptotic distribution of $\hat{\theta}_{2n}$ has no impact on that of $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n$. We give specific regularity conditions below and formally show this result in Appendix C. Let $\mathcal{L}_n(\phi) = \max_{\beta,\sigma^2} \mathrm{E}[\ln L_n(\theta)]$, where $\phi = (\lambda, \rho)'$.

Assumption 1. *(i) The elements of X_n are uniformly bounded constants, X_n has full column rank, and $\lim_{n\to\infty}\frac{1}{n}X_n'X_n$ exists and is nonsingular. (ii) W_n and M_n have zero diagonals, S_n and R_n are nonsingular, and the sequences of matrices $\{W_n\}$, $\{M_n\}$, $\{S_n^{-1}\}$ and $\{R_n^{-1}\}$ are bounded in both row and column sum norms.*

Assumption 2. *v_{ni}'s are i.i.d. with mean zero, variance σ_0^2 and $\mathrm{E}(|v_{ni}|^{4+\alpha}) < \infty$ for some $\alpha > 0$.*

Assumption 3. *(i) $\lim_{n\to\infty}\frac{1}{n}\frac{\partial^2\mathcal{L}_n(\phi_0)}{\partial\phi\partial\phi'}$ exists and is nonsingular. (ii) $\lim_{n\to\infty}\frac{1}{n}\,\mathrm{E}(\frac{\partial \ln L_n(0,\theta_{20})}{\partial\theta}\frac{\partial \ln L_n(0,\theta_{20})}{\partial\theta'})$ exists and is nonsingular. (iii) $\sqrt{n}(\hat{\theta}_{2n} - \theta_{20}) = O_p(1)$.*

Assumptions 1 and 2 are the same as those in Lee (2007), with additional conditions on M_n similar to those on W_n. In Assumption 2, v_{ni}'s are assumed to be homoskedastic but they can be non-normal. Assumption 3(i) guarantees the nonsingularity of $\lim_{n\to\infty}\frac{1}{n}\Omega_n$. If v_{ni}'s are normal, the information matrix equality that $\Omega_n = -\,\mathrm{E}(\frac{\partial \ln L_n(0,\theta_{20})}{\partial\theta}\frac{\partial \ln L_n(0,\theta_{20})}{\partial\theta'})$ holds, so Assumption 3(ii) holds under Assumption 3(i). Otherwise,

[8] For the test of $\rho_0 = 0$ in the SE model, the orthogonality condition holds. For the test of $\lambda_0 = 0$ in the SAR model and that of $\lambda_0 = \rho_0 = 0$ in the SARAR model, the orthogonality condition does not hold. However, by introducing the projection matrix $I_n - X_n(X_n'X_n)^{-1}X_n'$ in the statistic $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial\theta_1}$, which does not change the magnitude of $\frac{1}{\sqrt{n}}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial\theta_1}$ but properly takes into account the asymptotic impact of the initial estimate, a test statistic of the form (10) can still be derived.

[9] For the tests of $\rho_0 = 0$ or $\lambda_0 = 0$, the restricted QML estimator does not have a closed form, unlike the cases in Born and Breitung (2011). Thus the OPG variants of LM tests cannot be derived as in Born and Breitung (2011).

[10] See (22)–(26) of Appendix A for their explicit expressions.

$\frac{1}{n}\,\mathrm{E}(\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta}\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta'})$ generally involves the third and fourth moments of v_{ni}.[11] The $\sqrt{n}$-consistency of $\hat{\theta}_{2n}$ in Assumption 3(*iii*) can be seen from Kelejian and Prucha (1998), Liu et al. (2010) or Jin and Lee (2013) if it is, respectively, the spatial 2SLS estimator, the GMM estimator, or the QML estimator.

Proposition 1. *Under Assumptions 1–3, when* $\theta_{10} = 0$, $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n = \frac{1}{\sqrt{n}}\Upsilon_n + o_p(1)$.

As Υ_n is a vector of linear-quadratic forms, let $\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta} = \sum_{i=1}^n \xi_{ni}$, where ξ_{ni}'s are martingale differences as for a Ξ_n above. Correspondingly, $\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta} = \sum_{i=1}^n \hat{\xi}_{ni}$. Denote $\hat{\zeta}_{ni} = [I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\xi}_{ni}$ and $\hat{\varphi}_n = (\hat{\xi}_{n1}, \ldots, \hat{\xi}_{nn})'$. By (8), the OPG test is

$$\hat{\Upsilon}_n'\big([I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big)^{-1}\hat{\Upsilon}_n = \Big(\sum_{i=1}^n \hat{\zeta}_{ni}\Big)'\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\hat{\zeta}_{ni}'\Big)^{-1}\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\Big), \tag{12}$$

which is asymptotically $\chi^2(k_1)$ under regularity conditions in Proposition 2 below. This statistic can be computed by a regression package as the explained sum of squares of the regression of the constant 1 on $\hat{\zeta}_{ni}$. In (11), if $\hat{\theta}_{2n}$ is the restricted QML estimator $\tilde{\theta}_{2n}$ with the restriction $\theta_1 = 0$ imposed in the estimation, since $\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_2} = 0$, $\hat{\Upsilon}_n = \frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1}$ and hence (12) is the LM test statistic

$$\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1'}\big([I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big)^{-1}\frac{\partial \ln L_n(0,\tilde{\theta}_{2n})}{\partial \theta_1}$$

with a proper OPG variance estimate. This LM test statistic generalizes the use of OPG variants even the corresponding orthogonality condition on $\Omega_{n,12}$ converging to zero is not satisfied. We note that while $\frac{\partial L_n(0,\tilde{\theta}_{2n})}{\partial \theta_2}$ can be zero, it does not mean that the corresponding components in $\hat{\xi}_{ni}$ would be zero for all i. So the procedure of using a regression of 1 on $\hat{\zeta}_{ni}$ to construct the OPG variant of a test statistic remains the same regardless what consistent estimates are used. The only difference is their uses on the construction of the estimated $\hat{\xi}_{ni}$ for $i = 1, \cdots, n$. When the QML estimator for an SARAR model turns out to be computationally intensive, other computationally simple estimators such as the GMM and spatial 2SLS estimators can be used for the OPG test.[12]

Proposition 2. *Under Assumptions 1–3, when* $\theta_{10} = 0$, $\hat{\Upsilon}_n'([I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]')^{-1}\hat{\Upsilon}_n = (\sum_{i=1}^n \hat{\zeta}_{ni})'(\sum_{i=1}^n \hat{\zeta}_{ni}\hat{\zeta}_{ni}')^{-1}(\sum_{i=1}^n \hat{\zeta}_{ni}) \xrightarrow{d} \chi^2(k_1)$.

We apply the above statistics to the tests of $\rho_0 = 0$ and/or $\lambda_0 = 0$ in the SARAR model below.

2.1.1 Testing for $\rho_0 = 0$

For the test of whether the spatial error dependence $\rho_0 = 0$ in the presence of spatial lag dependence, we have $\theta = (\theta_1, \theta_2')'$, $\theta_1 = \rho$ and $\theta_2 = (\lambda, \beta', \sigma^2)'$. With the restriction $\rho_0 = 0$, (3)–(6) become

$$\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta} = \begin{pmatrix} \frac{1}{\sigma_0^2}V_n'M_nV_n \\ \frac{1}{\sigma_0^2}V_n'G_nV_n - \mathrm{tr}(G_n) + \frac{1}{\sigma_0^2}(G_nX_n\beta_0)'V_n \\ \frac{1}{\sigma_0^2}X_n'V_n \\ \frac{1}{2\sigma_0^4}V_n'V_n - \frac{n}{2\sigma_0^2} \end{pmatrix}.$$

[11] The explicit expression of $\frac{1}{n}\,\mathrm{E}(\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta}\frac{\partial \ln L_n(0,\theta_{20})}{\partial \theta'})$ is in Jin and Lee (2013).

[12] A concise case can be a high order SARAR model with several spatial weights matrices.

By (7), we define a matrix φ_n of martingale differences as

$$\varphi_n = \frac{1}{\sigma_0^2}[V_n \circ (\text{tril}(M_n^{(s)})V_n), V_n \circ \text{vec}_{\text{D}}(G_n) \circ V_n - \sigma_0^2 \text{vec}_{\text{D}}(G_n) + V_n \circ (\text{tril}(G_n^{(s)})V_n) + (G_n X_n \beta_0) \circ V_n,$$
$$X_n \circ (l_k' \otimes V_n), \frac{1}{2\sigma_0^2} V_n \circ V_n - \frac{1}{2} l_n],$$

where l_n is an $n \times 1$ vector of ones and $\otimes$ denotes the Kronecker product. By (22)–(26) in Appendix A,

$$\Omega_{n,12} \equiv -\text{E}\big(\frac{\partial^2 \ln L_n(0, \theta_{20})}{\partial \rho \partial \theta_2'}\big) = \big(\text{tr}(M_n^{(s)} G_n), 0_{1\times(k+1)}\big),$$

and

$$\Omega_{n,22} \equiv -\text{E}\big(\frac{\partial^2 \ln L_n(0, \theta_{20})}{\partial \theta_2 \partial \theta_2'}\big) = \begin{pmatrix} \frac{1}{\sigma_0^2}(G_n X_n \beta_0)' G_n X_n \beta_0 + \text{tr}(G_n^{(s)} G_n) & \frac{1}{\sigma_0^2}(G_n X_n \beta_0)' X_n & \frac{1}{\sigma_0^2}\text{tr}(G_n) \\ \frac{1}{\sigma_0^2} X_n' G_n X_n \beta_0 & \frac{1}{\sigma_0^2} X_n' X_n & 0 \\ \frac{1}{\sigma_0^2}\text{tr}(G_n) & 0 & \frac{n}{2\sigma_0^4} \end{pmatrix}.$$

Since $\lim_{n\to\infty} \frac{1}{n}\Omega_{n,12} \neq 0$, the orthogonality condition does not hold. Thus, for the LM test, it is necessary to properly take into account the impact of the restricted QML estimator to obtain an OPG variant. In this case, it is convenient to use the form of a $C(\alpha)$ statistic. Let $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$ be terms obtained by replacing θ_{20} with a $\sqrt{n}$-consistent estimator $\hat{\theta}_{2n}$ in, respectively, $\Omega_{n,12}$ and $\Omega_{n,22}$, and let $\hat{\varphi}_n$ be the estimate of φ_n by replacing V_n in φ_n with $V_n(\hat{\delta}_n)$ and θ_{20} with $\hat{\theta}_{2n}$, where $\hat{\delta}_n$, an estimator of δ, is a subvector of $\hat{\theta}_n = (0, \hat{\theta}_{2n}')'$. Then by Proposition 2,

$$\frac{\partial \ln L_n(0, \hat{\theta}_{2n})}{\partial \theta'}[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big([1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big)^{-1}[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\frac{\partial \ln L_n(0, \hat{\theta}_{2n})}{\partial \theta} \xrightarrow{d} \chi^2(1).$$

For $\hat{\theta}_{2n}$, the restricted model with $\rho = 0$ imposed is an SAR model, for which a least squares (LS) estimator is generally inconsistent. In the event that the QML estimator is computationally intensive for a large sample size, we may use a GMM estimator or simply the 2SLS estimator, which are $\sqrt{n}$-consistent and computationally simple.

2.1.2 Testing for $\lambda_0 = 0$

To test for only spatial lag dependence, denote $\theta = (\lambda, \theta_2')'$, where $\theta_2 = (\rho, \beta', \sigma^2)'$. Then by (3)–(6),

$$\frac{\partial \ln L_n(0, \theta_{20})}{\partial \theta} = \begin{pmatrix} \frac{1}{\sigma_0^2} V_n' \ddot{W}_n V_n + \frac{1}{\sigma_0^2}(R_n W_n X_n \beta_0)' V_n \\ \frac{1}{\sigma_0^2} V_n' H_n V_n - \text{tr}(H_n) \\ \frac{1}{\sigma_0^2} X_n' R_n' V_n \\ \frac{1}{2\sigma_0^4} V_n' V_n - \frac{n}{2\sigma_0^4} \end{pmatrix},$$

where $\ddot{W}_n = R_n W_n R_n^{-1}$. Using the formula in (7), we define a matrix of martingale differences

$$\varphi_n = \frac{1}{\sigma_0^2}[V_n \circ \text{vec}_{\text{D}}(\ddot{W}_n) \circ V_n - \sigma_0^2 \text{vec}_{\text{D}}(\ddot{W}_n) + V_n \circ (\text{tril}(\ddot{W}_n^{(s)})V_n) + (R_n W_n X_n \beta_0) \circ V_n,$$
$$V_n \circ \text{vec}_{\text{D}}(H_n) \circ V_n - \sigma_0^2 \text{vec}_{\text{D}}(H_n) + V_n \circ (\text{tril}(H_n^{(s)})V_n), (R_n X_n) \circ (l_k' \otimes V_n), \frac{1}{2\sigma_0^2} V_n \circ V_n - \frac{1}{2} l_n].$$

Imposing the restriction $\lambda_0 = 0$ in (22)–(26), we have

$$\Omega_{n,12} \equiv -\text{E}\big(\frac{\partial^2 \ln L_n(0, \gamma_0)}{\partial \lambda \partial \theta_2'}\big) = \big(\text{tr}(H_n^{(s)} \ddot{W}_n), \frac{1}{\sigma_0^2}(R_n W_n X_n \beta_0)' R_n X_n, 0\big),$$

and

$$\Omega_{n,22} \equiv -\mathrm{E}\Big(\frac{\partial^2 \ln L_n(0,\gamma_0)}{\partial\theta_2 \partial\theta_2'}\Big) = \begin{pmatrix} \mathrm{tr}(H_n^{(s)} H_n) & 0 & \frac{1}{\sigma_0^2}\mathrm{tr}(H_n) \\ 0 & \frac{1}{\sigma_0^2} X_n' R_n' R_n X_n & 0 \\ \frac{1}{\sigma_0^2}\mathrm{tr}(H_n) & 0 & \frac{n}{2\sigma_0^4} \end{pmatrix}.$$

Note that the orthogonality condition $\lim_{n\to\infty} \frac{1}{n}\Omega_{n,12} = 0$ does not hold. With a $\sqrt{n}$ consistent estimator $\hat{\theta}_{2n}$, let $\hat{\varphi}_n$, $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$ be the estimates of φ_n, $\Omega_{n,12}$ and $\Omega_{n,22}$ by $\hat{\theta}_{2n}$ as above. The OPG test is

$$\frac{\partial \ln L_n(0,\hat{\theta}_{2n})}{\partial\theta'}[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big([1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big)^{-1}[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\frac{\partial \ln L_n(0,\hat{\theta}_{2n})}{\partial\theta},$$

which is asymptotically $\chi^2(1)$ distributed.[13]

2.2 Tests under unknown heteroskedasticity

If the disturbances v_{ni}'s are independent but with different variances σ_{ni}^2's, the scores in (3)–(6) for the SARAR model do not have zero expected values. This is so because the ML estimate of the SARAR model uses quadratic moments based on i.i.d. disturbances. Under unknown heteroskedasticity, the likelihood function would be misspecified and it would not even be a quasi-likelihood for the SARAR model. This misspecification shows up in the scores having non-zero means. In Lin and Lee (2010), linear and quadratic moments are used for consistent estimation, where quadratic matrices are modified to have zero diagonals so that suggested quadratic moments are proper moments for estimation. In a recent paper, Liu and Yang (2015) modify the scores to have zero means such that consistent estimators under unknown heteroskedasticity can be derived by solving such modified scores. As we have focused our attention on score-based OPG tests in previous sections, we consider the use of modified scores in Liu and Yang (2015) for testing problems, and derive robust OPG tests for spatial dependence under unknown heteroskedasticity. For SARAR models, the modified scores are proper moments, so suggested tests are essentially moment test statistics (Tauchen, 1985; Newey, 1985).[14]

The nonzero expected values of the scores (3)–(6) are due to quadratic forms $V_n' A_n V_n$ for an $n \times n$ matrix A_n with a nonzero diagonal and its mean is not $\sigma_0^2\,\mathrm{tr}(A_n)$. To make a quadratic form have zero expected value, we can modify it to be $V_n'[A_n - \mathrm{diag}(A_n)]V_n$ (Lin and Lee, 2010; Kelejian and Prucha, 2010), where $\mathrm{diag}(A_n)$ denotes a diagonal matrix formed by the diagonal elements of A_n.[15] By (3)–(5), tests can be based on the statistics[16]

$$g_{n\lambda}(\delta) = V_n'(\delta)[\ddot{G}_n(\phi) - \mathrm{diag}(\ddot{G}_n(\phi))]V_n(\delta) + (R_n(\rho)G_n(\lambda)X_n\beta)'V_n(\delta), \tag{13}$$

[13]Since the restricted model with $\lambda = 0$ is an SE model, we may even use the LS estimator of β for the construction of the OPG test in this case.

[14]A special feature with modified scores as moments is that the number of moments is just identifiable for the parameters of the unrestricted model. If there were overidentified moment conditions, we suggest the use of gradient tests in Newey (1985). That approach will be considered in a subsequent section.

[15]Note that $\mathrm{E}(V_n' A_n V_n) = \mathrm{tr}(A_n\Sigma_n) = \sum_{i=1}^n a_{n,ii}\sigma_{ni}^2$. If we would like $\mathrm{E}(V_n' A_n V_n)$ to be zero for arbitrary σ_{ni}^2's, then $a_{n,ii}$'s should all be zero. Alternatively, since $\mathrm{tr}(A_n\Sigma_n) = \mathrm{E}(V_n'\,\mathrm{diag}(A_n)V_n)$, we can rewrite $\mathrm{E}(V_n' A_n V_n) = \mathrm{tr}(A_n\Sigma_n)$ as $\mathrm{E}[V_n'(A_n - \mathrm{diag}(A_n))V_n] = 0$. Then we also end up with a matrix $A_n - \mathrm{diag}(A_n)$ with a zero diagonal. The latter motivation is discussed in Kelejian and Prucha (2010, p. 56, Section 3.1).

[16]Since σ^2 is not an interesting parameter, we can ignore the score in (6). So we focus on the subvector δ of θ in this subsection for

$$g_{n\rho}(\delta) = V_n'(\delta)[H_n(\rho) - \mathrm{diag}(H_n(\rho))]V_n(\delta), \tag{14}$$

$$g_{n\beta}(\delta) = X_n' R_n'(\rho) V_n(\delta), \tag{15}$$

where $G_n(\lambda) = W_n S_n^{-1}(\lambda)$, $\ddot{G}_n(\phi) = R_n(\rho)G_n(\lambda)R_n^{-1}(\rho)$, and $H_n(\rho) = M_n R_n^{-1}(\rho)$. The $g_{n\lambda}(\delta)$ and $g_{n\rho}(\delta)$ are modified, respectively, from the derivatives of the QML function with respect to λ and ρ. At δ_0, they are linear-quadratic forms of V_n, so their OPG variance estimates can be derived similarly as before by the use of martingale differences ξ_{ni}'s. For matrices A_{ni}'s with zero diagonals, the variance of a vector of linear-quadratic forms $\Xi_n = [V_n' A_{n1} V_n + b_{n1}' V_n, \ldots, V_n' A_{np} V_n + b_{np}' V_n]'$ is $\mathrm{E}(\varphi_n' \varphi_n)$, where $\varphi_n = (\xi_{n1}, \ldots, \xi_{nn})'$ is an $n \times p$ matrix of martingale differences and ξ_{ni} is a $p \times 1$ vector, and explicitly,

$$\varphi_n = [V_n \circ (\mathrm{tril}(A_{n1}^{(s)})V_n) + b_{n1} \circ V_n, \ldots, V_n \circ (\mathrm{tril}(A_{np}^{(s)})V_n) + b_{np} \circ V_n]. \tag{16}$$

This variance can be estimated by $\hat{\varphi}_n' \hat{\varphi}_n$, where $\hat{\varphi}_n$ is obtained by replacing V_n in φ_n by $V_n(\hat{\delta}_n)$ with a consistent estimator $\hat{\delta}_n$ and also any δ_0 in A_{ni}'s and b_{ni}'s by $\hat{\delta}_n$.

For generality, we consider the test of whether the true value δ_{10} for a $k_1 \times 1$ subvector δ_1 of $\delta = (\delta_1', \delta_2')'$ is zero or not. Let $g_{n1}(\delta)$ be the subvector in $(g_{n\lambda}(\delta), g_{n\rho}(\delta), g_{n\beta}'(\delta))'$ that correspond to derivatives with respect to δ_1. Define $g_{n2}(\delta)$ similarly and denote $g_n(\delta) = (g_{n1}'(\delta), g_{n2}'(\delta))'$. As in the last subsection, assume that $g_n(0, \delta_{20}) = \frac{1}{n}\sum_{i=1}^n \xi_{ni}$, where ξ_{ni}'s are martingale differences. Let $\Omega_{n,12} = -\mathrm{E}(\frac{\partial g_{n1}(0,\delta_{20})}{\partial \delta_2'})$, $\Omega_{n,22} = -\mathrm{E}(\frac{\partial g_{n2}(0,\delta_{20})}{\partial \delta_2'})$,[17] and $\Sigma_n = \mathrm{diag}(\sigma_{n1}^2, \ldots, \sigma_{nn}^2)$ be a diagonal matrix formed by the unknown variances σ_{ni}^2's. For any $\sqrt{n}$-consistent estimator $\hat{\delta}_{2n}$ of δ_{20}, let $\hat{V}_n = (\hat{v}_{ni}) = V_n(0, \hat{\delta}_{2n})$ and $\hat{\Sigma}_n = \mathrm{diag}(\hat{v}_{n1}^2, \ldots, \hat{v}_{nn}^2)$. Replacing Σ_n in $\Omega_{n,12}$ and $\Omega_{n,22}$ by $\hat{\Sigma}_n$ and δ_{20} by $\hat{\delta}_{2n}$ yields their estimates $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$. Denote $\hat{\zeta}_{ni} = (I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1})\hat{\xi}_{ni}$ as before, where $\hat{\xi}_{ni}$'s are estimated ξ_{ni}'s with $\hat{\delta}_n = (0, \hat{\delta}_{2n}')'$, and

$$\hat{\Upsilon}_n = [I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]g_n(0, \hat{\delta}_{2n}). \tag{17}$$

Under regularity conditions, in Proposition 3 below, $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n = \frac{1}{\sqrt{n}}\Upsilon_n + o_p(1)$, where $\Upsilon_n = [I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]g_n(0, \delta_{20})$, so the asymptotic distribution of $\hat{\delta}_{2n}$ does not have an impact on that of $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n$.

Assumption 4. *v_{ni}'s are independent with mean zero and variances σ_{ni}^2's, and $\sup_n \sup_{1\le i\le n} \mathrm{E}(|v_{ni}|^{4+\alpha}) < \infty$ for some $\alpha > 0$.*

Assumption 5. *(i) $\lim_{n\to\infty} \frac{1}{n}\mathrm{E}(\frac{\partial g_n(0,\delta_{20})}{\partial \delta'})$ and $\lim_{n\to\infty}\frac{1}{n}\mathrm{E}[g_n(0,\delta_{20})g_n'(0,\delta_{20})]$ exist and are nonsingular. (ii) $\sqrt{n}(\hat{\delta}_{2n} - \delta_{20}) = O_p(1)$.*

Assumption 4 on heteroskedasticity is the same as that in Lin and Lee (2010). $\lim_{n\to\infty}\frac{1}{n}\mathrm{E}(\frac{\partial g_n(0,\delta_{20})}{\partial \delta'})$ and $\lim_{n\to\infty}\frac{1}{n}\mathrm{E}[g_n(0,\delta_{20})g_n'(0,\delta_{20})]$ are assumed to be nonsingular in Assumption 5(i) so that the asymptotic chi-squared distribution of our test statistic has the number of degrees of freedom equal to that of restrictions. The

analysis. Instead of (13)–(15), we may also first concentrate away β and then correct the scores for λ and ρ (Liu and Yang, 2015). But that generates more complicated OPG tests for the SARAR model. It is of interest in a future research to compare the finite sample performance of the two methods by high order expansions and Monte Carlo experiments.

[17] Their explicit expressions are in (27)–(32) of Appendix A.

$\sqrt{n}$-consistent estimator $\hat{\delta}_{2n}$ in Assumption 5(ii) is usually derived from the GMM estimation such as that in Kelejian and Prucha (2010) or Lin and Lee (2010).

Proposition 3. *Under Assumptions 1, 4 and 5, when* $\delta_{10} = 0$, $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n = \frac{1}{\sqrt{n}}\Upsilon_n + o_p(1)$.

The statistic in (12) has the OPG form which can be the explained sum of squares from a regression of the constant 1 on $\hat{\zeta}_{ni}$. This statistic is asymptotically chi-squared distributed.

Proposition 4. *Under Assumptions 1, 4 and 5, when* $\delta_{10} = 0$, $\hat{\Upsilon}_n'\big([I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big)^{-1}\hat{\Upsilon}_n = (\sum_{i=1}^n \hat{\zeta}_{ni})'(\sum_{i=1}^n \hat{\zeta}_{ni}\hat{\zeta}_{ni}')^{-1}(\sum_{i=1}^n \hat{\zeta}_{ni}) \xrightarrow{d} \chi^2(k_1)$.

2.2.1 Testing for $\rho_0 = 0$

For the test of $\rho_0 = 0$, denote $\delta = (\rho, \delta_2')'$, where $\delta_2 = (\lambda, \beta')'$. Then

$$g_n(0, \delta_{20}) = \begin{pmatrix} V_n'M_nV_n \\ V_n'[G_n - \operatorname{diag}(G_n)]V_n + (G_nX_n\beta_0)'V_n \\ X_n'V_n \end{pmatrix}.$$

To estimate the variance of $g_n(0, \delta_{20})$, define the matrix φ_n of martingale differences to be

$$\varphi_n = [V_n \circ (\operatorname{tril}(M_n^{(s)})V_n), V_n \circ (\operatorname{tril}(G_n^{(s)} - \operatorname{diag}(G_n^{(s)}))V_n) + (G_nX_n\beta_0) \circ V_n, X_n \circ (l_k' \otimes V_n)].$$

Imposing the restriction $\rho_0 = 0$ in (27)–(32) of Appendix A, we have

$$\Omega_{n,12} \equiv -\operatorname{E}\big(\frac{\partial g_{n1}(0, \delta_{20})}{\partial \delta_2'}\big) = [\operatorname{tr}(M_n^{(s)}G_n\Sigma_n), 0_{1\times k}],$$

and

$$\Omega_{n,22} \equiv -\operatorname{E}\big(\frac{\partial g_{n2}(0, \delta_{20})}{\partial \delta_2'}\big) = \begin{pmatrix} \operatorname{tr}[(G_n - \operatorname{diag}(G_n))^sG_n\Sigma_n] + (G_nX_n\beta_0)'G_nX_n\beta_0 & (G_nX_n\beta_0)'X_n \\ X_n'G_nX_n\beta_0 & X_n'X_n \end{pmatrix}.$$

Let $\hat{\varphi}_n$, $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$ be estimates of φ_n, $\Omega_{n,12}$ and $\Omega_{n,22}$ with a $\sqrt{n}$-consistent estimator $\hat{\delta}_{2n}$ of δ_{20}. The OPG test is

$$g_n'(0, \hat{\delta}_{2n})[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big([1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\varphi}_n'\hat{\varphi}_n[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]'\big)^{-1}[1, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]g_n(0, \hat{\delta}_{2n}),$$

which follows the asymptotic $\chi^2(1)$ distribution.

2.2.2 Testing for $\lambda_0 = 0$

To test for $\lambda_0 = 0$, let $\delta = (\lambda, \delta_2')'$ and $\delta_2 = (\rho, \beta')'$. Then,

$$g_n(0, \delta_{20}) = \begin{pmatrix} V_n'[\ddot{W}_n - \operatorname{diag}(\ddot{W}_n)]V_n + (R_nW_nX_n\beta_0)'V_n \\ V_n'[H_n - \operatorname{diag}(H_n)]V_n \\ X_n'R_n'V_n \end{pmatrix},$$

and

$$\varphi_n = [V_n \circ (\text{tril}(\ddot{W}_n^{(s)} - \text{diag}(\ddot{W}_n^{(s)}))V_n) + (R_n W_n X_n \beta_0) \circ V_n, V_n \circ (\text{tril}(H_n^{(s)} - \text{diag}(H_n^{(s)}))V_n), (R_n X_n) \circ (l_k' \otimes V_n)].$$

Imposing the restriction $\lambda_0 = 0$ in (27)–(32), we have

$$\Omega_{n,12} \equiv -\mathrm{E}(\frac{\partial g_{n1}(0, \delta_{20})}{\partial \delta_2'}) = [\text{tr}[(\ddot{W}_n - \text{diag}(\ddot{W}_n))^{(s)} H_n \Sigma_n], (R_n W_n X_n \beta_0)' R_n X_n],$$

and

$$\Omega_{n,22} \equiv -\mathrm{E}(\frac{\partial g_{n2}(0, \delta_{20})}{\partial \delta_2'}) = \begin{pmatrix} \text{tr}[(H_n - \text{diag}(H_n))^s H_n \Sigma_n] & 0 \\ 0 & X_n' R_n' R_n X_n \end{pmatrix}.$$

With these expressions, we can derive the OPG test statistic with a $\sqrt{n}$-consistent estimator $\hat{\delta}_{2n}$ of δ_{20} as before.

3 Gradient-based OPG tests in GMM

The tests considered in the previous section are based on QML scores. In this section, we consider gradient-based OPG tests in the GMM framework. The gradient test is general in the sense that the number of moments involved in estimation of the unrestricted model can be over-identified.

3.1 Tests under homoskedasticity

For the SARAR model (1), an alternative estimation method in place of the QML is a GMM method. In that framework, the corresponding LM test is a gradient test. If the disturbances v_{ni}'s are homoskedastic, consider the GMM estimation with the moment vector

$$g_n(\delta) = [V_n'(\delta) P_{n1} V_n(\delta), \ldots, V_n'(\delta) P_{n,k_p} V_n(\delta), V_n'(\delta) F_n]', \tag{18}$$

where P_{ni} for $i = 1, \ldots, k_p$ are $n \times n$ nonstochastic matrices with zero traces, and F_n is an $n \times k_f$ matrix of instrumental variables (IV) such that $k_p + k_f \geq k + 2$. The quadratic matrices P_{ni}'s can be constructed from the spatial weights matrices W_n and M_n, and F_n consists of independent columns of X_n, $W_n X_n$, $W_n^2 X_n$ and so on.[18] Since $g_n(\delta_0)$ is a vector of linear-quadratic forms, its variance can be computed with the OPG formula. Let

$$\begin{aligned}\varphi_n = [&V_n \circ \text{vec}_{\mathrm{D}}(P_{n1}) \circ V_n - \sigma_0^2 \,\text{vec}_{\mathrm{D}}(P_{n1}) + V_n \circ (\text{tril}(P_{n1}^{(s)})V_n), \ldots, \\ &V_n \circ \text{vec}_{\mathrm{D}}(P_{n,k_p}) \circ V_n - \sigma_0^2 \,\text{vec}_{\mathrm{D}}(P_{n,k_p}) + V_n \circ (\text{tril}(P_{n,k_p}^{(s)})V_n), F_n \circ (l_{k_f}' \otimes V_n)].\end{aligned}$$

Then the variance of $g_n(\delta_0)$ is $\Delta_n = \mathrm{E}(\varphi_n' \varphi_n)$. Let $\dot{\delta}_n$ be an initial GMM estimator. An estimate $\dot{\Delta}_n$ of Δ_n is $\dot{\varphi}_n' \dot{\varphi}_n$, where $\dot{\varphi}_n$ is derived by replacing δ_0 in φ_n with $\dot{\delta}_n$, V_n with $V_n(\dot{\delta}_n)$, σ_0^2 with $\frac{1}{n} V_n'(\dot{\delta}_n) V_n(\dot{\delta}_n)$, and any unknown parameters in P_{ni}'s and F_n with corresponding estimates. With $\dot{\Delta}_n$, the criterion function for the optimal GMM estimator is $g_n'(\delta) \dot{\Delta}_n^{-1} g_n(\delta)$. In particular, for the test of whether a $k_1 \times 1$ subvector δ_{10} of δ_0 is zero, a

[18] See Lee (2007) for details.

constrained optimal GMM estimator $\tilde{\delta}_n = (0, \tilde{\delta}'_{2n})'$ can be obtained from the minimization of $g'_n(0, \delta_2)\dot{\Delta}_n^{-1} g_n(0, \delta_2)$. Let $\mathcal{G}_{nj}(\delta) = \frac{\partial g_n(\delta)}{\partial \delta'_j}$ for $j = 1, 2$ and $\mathcal{G}_n(\delta) = \frac{\partial g_n(\delta)}{\partial \delta'}$. Based on the asymptotic distribution of $\mathcal{G}'_n(\tilde{\delta})\dot{\Delta}_n^{-1} g_n(\tilde{\delta}_n)$, the gradient test statistic

$$g'_n(\tilde{\delta}_n)\dot{\Delta}_n^{-1}\mathcal{G}_n(\tilde{\delta}_n)[\mathcal{G}'_n(\tilde{\delta}_n)\dot{\Delta}_n^{-1}\mathcal{G}_n(\tilde{\delta}_n)]^{-1}\mathcal{G}'_n(\tilde{\delta}_n)\dot{\Delta}_n^{-1} g_n(\tilde{\delta}_n) \tag{19}$$

is asymptotically $\chi^2(k_1)$ distributed. It is emphasized in Newey and West (1987) that the above formula must be evaluated at the constrained optimal GMM estimator $\tilde{\delta}_n$. To use any $\sqrt{n}$-consistent estimator $\hat{\delta}_{2n}$ of δ_{20}, we may use the $C(\alpha)$-type statistic. Denote $\mathbb{G}_{nj} = \mathrm{E}[\mathcal{G}_{nj}(\delta_0)]$ for $j = 1, 2$, and $\mathbb{G}_n = \mathrm{E}[\mathcal{G}_n(\delta_0)]$.[19] Let $\hat{\mathbb{G}}_n$ be an estimate of $\mathbb{G}_n$, obtained by replacing δ_0 in $\mathbb{G}_n$ by $\hat{\delta}_n = (0, \hat{\delta}'_{2n})'$ and σ_0^2 by $\frac{1}{n}V'_n(\hat{\delta}_n)V_n(\hat{\delta}_n)$. Correspondingly we have estimates $\hat{\mathbb{G}}_{n1}$ and $\hat{\mathbb{G}}_{n2}$ of $\mathbb{G}_{n1}$ and $\mathbb{G}_{n2}$. Denote $\Omega_{n,ij} = \mathbb{G}'_{ni}\Delta_n^{-1}\mathbb{G}_{nj}$ for $j = 1, 2$, $\Omega_n = \mathbb{G}'_n\Delta_n^{-1}\mathbb{G}_n$, $\hat{\Omega}_{n,ij} = \hat{\mathbb{G}}'_{ni}\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_{nj}$ for $j = 1, 2$, and $\hat{\Omega}_n = \hat{\mathbb{G}}'_n\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_n$, where $\hat{\Delta}_n = \hat{\varphi}'_n\hat{\varphi}_n$ and $\hat{\varphi}_n$ is obtained using $\hat{\delta}_n$. We can consider the $C(\alpha)$-type statistic[20]

$$\hat{\Upsilon}_n = [I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\mathbb{G}}'_n\hat{\Delta}_n^{-1} g_n(\hat{\delta}_n).$$

Under regularity conditions, Proposition 5 gives $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n = \frac{1}{\sqrt{n}}\Upsilon_n + o_p(1)$, where $\Upsilon_n = [I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\mathbb{G}'_n\Delta_n^{-1} g_n(\delta_0)$. Let $\hat{\zeta}_{ni} = [I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\mathbb{G}}'_n\hat{\Delta}_n^{-1}\hat{\xi}_{ni}$, where $\hat{\xi}_{ni}$'s are approximate martingale differences in $\hat{\varphi}_n = (\hat{\xi}_{n1}, \ldots, \hat{\xi}_{nn})'$. Then the OPG test statistic for $\delta_{10} = 0$ is

$$\hat{\Upsilon}'_n\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\hat{\zeta}'_{ni}\Big)^{-1}\hat{\Upsilon}_n = \Big(\sum_{i=1}^n \hat{\zeta}_{ni}\Big)'\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\hat{\zeta}'_{ni}\Big)^{-1}\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\Big). \tag{20}$$

As $\hat{\Delta}_n = \sum_{i=1}^n \hat{\xi}_{ni}\hat{\xi}'_{ni}$ and $[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\Omega}_n[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]' = \hat{\Omega}_{n,11} - \hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21}$, this test statistic is equal to

$$\hat{\Upsilon}'_n(\hat{\Omega}_{n,11} - \hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21})^{-1}\hat{\Upsilon}_n = g'_n(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_n\hat{\Omega}_n^{-1}\hat{\mathbb{G}}'_n\hat{\Delta}_n^{-1} g_n(\hat{\delta}_n) - g'_n(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_{n2}\hat{\Omega}_{n,22}^{-1}\hat{\mathbb{G}}'_{n2}\hat{\Delta}_n^{-1} g_n(\hat{\delta}_n), \tag{21}$$

where the equality is shown in the proof of Proposition 5. If $\hat{\delta}_n = (0, \hat{\delta}'_{2n})'$ is the restricted GMM estimator $\tilde{\delta}_n = (0, \tilde{\delta}'_{2n})'$ from the minimization of $g'_n(0, \delta_2)\dot{\Delta}_n^{-1} g_n(0, \delta_2)$, then $\mathcal{G}'_{n2}(\hat{\delta}_{2n})\dot{\Delta}_n^{-1} g_n(\hat{\delta}_n) = 0$ and $\frac{1}{\sqrt{n}}\hat{\mathbb{G}}'_{n2}\hat{\Delta}_n^{-1} g_n(\hat{\delta}_n) = o_p(1)$; so the statistic is asymptotically equivalent to

$$g'_n(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_n\hat{\Omega}_n^{-1}\hat{\mathbb{G}}'_n\hat{\Delta}_n^{-1} g_n(\hat{\delta}_n),$$

and also asymptotically equivalent to the gradient test statistic in (19).

Assumption 6. *(i) P_{ni} for $i = 1, \ldots, p$ have zero traces. (ii) $\lim_{n\to\infty}\frac{1}{n}\Delta_n$ exists and is nonsingular, and $\lim_{n\to\infty}\frac{1}{n}\mathbb{G}_n$ exists and has full column rank. (iii) $\sqrt{n}(\hat{\delta}_{2n} - \delta_{20}) = O_p(1)$.*

When P_{ni}'s have zero traces, $g_n(\delta)$ is a valid moment vector for the SARAR model with homoskedastic disturbances. The nonsingularity of $\lim_{n\to\infty}\frac{1}{n}\Delta_n$ is needed for the formulation of the optimal GMM. The full column rank assumption on $\lim_{n\to\infty}\frac{1}{n}\mathbb{G}_n$ guarantees the $\sqrt{n}$-consistency of a GMM estimator.

[19] The explicit expression of $\mathbb{G}_n$ is in (33) of Appendix A.

[20] Alternatively, $\mathcal{G}_n(\hat{\delta}_n)$, $\mathcal{G}_{n1}(\hat{\delta}_n)$ and $\mathcal{G}_{n2}(\hat{\delta}_n)$ can be used in place of $\hat{\mathbb{G}}_n$, $\hat{\mathbb{G}}_{n1}$ and $\hat{\mathbb{G}}_{n2}$ in $\hat{\Upsilon}_n$, which will generate an asymptotically equivalent test statistic for $\delta_{10} = 0$. The usage of $\hat{\mathbb{G}}_n$, $\hat{\mathbb{G}}_{n1}$ and $\hat{\mathbb{G}}_{n2}$ is consistent with that of similar terms in score-based OPG tests in the last section.

Proposition 5. *Under Assumptions 1, 2 and 6, when* $\delta_{10} = 0$, $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n = \frac{1}{\sqrt{n}}\Upsilon_n + o_p(1)$, *and the gradient-based OPG test statistic*

$$\Big(\sum_{i=1}^{n}\hat{\zeta}_{ni}\Big)'\Big(\sum_{i=1}^{n}\hat{\zeta}_{ni}\hat{\zeta}_{ni}'\Big)^{-1}\Big(\sum_{i=1}^{n}\hat{\zeta}_{ni}\Big) = g_n'(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_n\hat{\Omega}_n^{-1}\hat{\mathbb{G}}_n'\hat{\Delta}_n^{-1}g_n(\hat{\delta}_n) - g_n'(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_{n2}\hat{\Omega}_{n,22}^{-1}\hat{\mathbb{G}}_{n2}'\hat{\Delta}_n^{-1}g_n(\hat{\delta}_n) \xrightarrow{d} \chi^2(k_1).$$

3.1.1 Testing for $\rho_0 = 0$

For the test of $\rho_0 = 0$, we have $\delta_1 = \rho$ and $\delta_2 = (\lambda, \beta')'$. Then $\mathbb{G}_{n1} = [\sigma_0^2 \operatorname{tr}(P_{n1}^{(s)}M_n), \ldots, \sigma_0^2 \operatorname{tr}(P_{n,k_p}^{(s)}M_n), 0_{1\times k_f}]'$ and

$$\mathbb{G}_{n2} = -\begin{pmatrix} \sigma_0^2 \operatorname{tr}(P_{n1}^{(s)}G_n) & 0 \\ \vdots & \vdots \\ \sigma_0^2 \operatorname{tr}(P_{n,k_p}^{(s)}G_n) & 0 \\ F_n'G_nX_n\beta_0 & F_n'X_n \end{pmatrix}.$$

With these expressions, the test statistic can be derived as above.

3.1.2 Testing for $\lambda_0 = 0$

To test for $\lambda_0 = 0$, denote $\delta_1 = \lambda$ and $\delta_2 = (\rho, \beta')'$. Then $\mathbb{G}_{n1} = [\sigma_0^2 \operatorname{tr}(P_{n1}^{(s)}\ddot{W}_n), \ldots, \sigma_0^2 \operatorname{tr}(P_{n,k_p}^{(s)}\ddot{W}_n), (R_nW_nX_n\beta_0)'F_n]'$ and

$$\mathbb{G}_{n2} = -\begin{pmatrix} \sigma_0^2 \operatorname{tr}(P_{n1}^{(s)}H_n) & 0 \\ \vdots & \vdots \\ \sigma_0^2 \operatorname{tr}(P_{n,k_p}^{(s)}H_n) & 0 \\ 0 & F_n'R_nX_n \end{pmatrix}.$$

3.2 Tests under unknown heteroskedasticity

If v_{ni}'s are heteroskedastic, the quadratic matrices P_{ni}'s for the moment vector $g_n(\delta)$ in (18) will be required to have zero diagonals so that $g_n(\delta)$ is a valid moment vector (Lin and Lee, 2010; Kelejian and Prucha, 2010). Then we can define a matrix φ_n of martingale differences

$$\varphi_n = [V_n \circ (\operatorname{tril}(P_{n1}^{(s)})V_n), \ldots, V_n \circ (\operatorname{tril}(P_{n,k_p}^{(s)})V_n), F_n \circ (l_{k_f}' \otimes V_n)],$$

and the variance of $g_n(\theta_0)$ is $\mathrm{E}(\varphi_n'\varphi_n)$. With a $\sqrt{n}$-consistent estimator $\hat{\delta}_{2n}$ of δ_{20}, let $\hat{\mathbb{G}}_n$, $\hat{\mathbb{G}}_{n1}$ and $\hat{\mathbb{G}}_{n2}$ be estimates of, respectively, $\mathbb{G}_n$, $\mathbb{G}_{n1}$ and $\mathbb{G}_{n2}$.[21] Correspondingly, $\hat{\xi}_{ni}$'s, $\hat{\Delta}_n$, $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$ have expressions similar to those in the last subsection. Then OPG test statistics in (20) and (21) are asymptotically chi-squared distributed as in Proposition 6.

Assumption 7. *(i)* P_{ni} *for* $i = 1, \ldots, p$ *have zero diagonals. (ii)* $\lim_{n\to\infty} \frac{1}{n}\Delta_n$ *exists and is nonsingular, and* $\lim_{n\to\infty} \frac{1}{n}\mathbb{G}_n$ *exists and has full column rank. (iii)* $\sqrt{n}(\hat{\delta}_{2n} - \delta_{20}) = O_p(1)$.

[21] The explicit expressions of $\mathbb{G}_n$, $\mathbb{G}_{n1}$ and $\mathbb{G}_{n2}$ are in (34) of Appendix A.

Proposition 6. *Under Assumptions 1, 4 and 7, when* $\delta_{10} = 0$, $\frac{1}{\sqrt{n}}\hat{\Upsilon}_n = \frac{1}{\sqrt{n}}\Upsilon_n + o_p(1)$, *and the gradient-based OPG test statistic*

$$\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\Big)'\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\hat{\zeta}'_{ni}\Big)^{-1}\Big(\sum_{i=1}^n \hat{\zeta}_{ni}\Big) = g'_n(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_n\hat{\Omega}_n^{-1}\hat{\mathbb{G}}'_n\hat{\Delta}_n^{-1}g_n(\hat{\delta}_n) - g'_n(\hat{\delta}_n)\hat{\Delta}_n^{-1}\hat{\mathbb{G}}_{n2}\hat{\Omega}_{n,22}^{-1}\hat{\mathbb{G}}'_{n2}\hat{\Delta}_n^{-1}g_n(\hat{\delta}_n) \xrightarrow{d} \chi^2(k_1).$$

3.2.1 Testing for $\rho_0 = 0$

For the test of $\rho_0 = 0$, $\delta_1 = \rho$ and $\delta_2 = (\lambda, \beta')'$. Then $\mathbb{G}_{n1} = [\mathrm{tr}(P_{n1}^{(s)}M_n\Sigma_n), \ldots, \mathrm{tr}(P_{n,k_p}^{(s)}M_n\Sigma_n), 0_{1\times k_f}]'$ and

$$\mathbb{G}_{n2} = -\begin{pmatrix} \mathrm{tr}(P_{n1}^{(s)}G_n\Sigma_n) & 0 \\ \vdots & \vdots \\ \mathrm{tr}(P_{n,k_p}^{(s)}G_n\Sigma_n) & 0 \\ F'_nG_nX_n\beta_0 & F'_nX_n \end{pmatrix}.$$

3.2.2 Testing for $\lambda_0 = 0$

For the test of $\lambda_0 = 0$, $\delta_1 = \lambda$ and $\delta_2 = (\rho, \beta')'$. Then $\mathbb{G}_{n1} = [\mathrm{tr}(P_{n1}^{(s)}\ddot{W}_n\Sigma_n), \ldots, \mathrm{tr}(P_{n,k_p}^{(s)}\ddot{W}_n\Sigma_n), (R_nW_nX_n\beta_0)'F_n]'$ and

$$\mathbb{G}_{n2} = -\begin{pmatrix} \mathrm{tr}(P_{n1}^{(s)}H_n\Sigma_n) & 0 \\ \vdots & \vdots \\ \mathrm{tr}(P_{n,k_p}^{(s)}H_n\Sigma_n) & 0 \\ 0 & F'_nR_nX_n \end{pmatrix}.$$

4 Monte Carlo

In this section, we conduct some Monte Carlo experiments to investigate the finite sample performance of the OPG tests for $\rho_0 = 0$ and $\lambda_0 = 0$ in the SARAR model (1).

Two row-normalized spatial weights matrices W_n are considered in the data generating process (DGP): one is based on the queen criterion and the other is based on the rook criterion. We set $M_n = W_n$. The exogenous variable matrix X_n contains three variables: the intercept term, a variable x_2 independently drawn from the standard normal distribution, and a third one x_3 independently drawn from the distribution $\chi^2(2)/2$. The true value β_0 of β is set to $(1,1,1)'$. In the homoskedastic case, the disturbances are independent draws from either the normal distribution $c_0N(0,1)$ or the normalized chi-squared distribution $c_0[\chi^2(2)-2]/2$, where c_0 is chosen such that $R^2 \equiv \mathrm{var}(x_{ni}\beta_0)/[\mathrm{var}(x_{ni}\beta_0)+\sigma_0^2]$ is 0.4 or 0.8. In the heteroskedastic case, we set $v_{ni} = c_0|x_{3n,i}|N(0,1)$ or $v_{ni} = c_0|x_{3n,i}|[\chi^2(2)-2]/2$, where $x_{3n,i}$ is the ith observation of the third exogenous variable, and c_0 is chosen such that $R^2 \equiv \mathrm{var}(x_{ni}\beta_0)/[\mathrm{var}(x_{ni}\beta_0)+\bar{\sigma}_n^2]$ is 0.4 or 0.8, where $\bar{\sigma}_n^2$ is the average variance $\frac{1}{n}\sum_{i=1}^n \sigma_{ni}^2$. The nominal size of tests is set to 0.05. To investigate power of tests for $\rho_0 = 0$, ρ_0 in the DGP is set to 0.2, 0.4, 0.6 or 0.8, while λ_0 is either 0.4 or 0.8. For power of tests for $\lambda_0 = 0$, similarly values are taken. Two sample sizes 144 and 400 are considered. The number of Monte Carlo repetitions is $1,000$.

To compute score-based OPG tests, the SAR and SE models are estimated by several methods.

(1) For the estimation of the SAR model:

(1.1) In the homoskedastic case, we consider the QML, 2SLS and two GMM estimators. The 2SLS estimator uses the IV matrix $[X_n, W_nX_{n1}, W_n^2X_{n1}]$, where X_{n1} is the submatrix of X_n that contains all non-constant variables.[22] The first GMM estimator, denoted by GMM1, is based on the moment vector $[V_n'W_nV_n, V_n'(W_n^2 - \mathrm{tr}(W_n^2)I_n/n)V_n, V_n'(X_n, W_nX_{n1}, W_n^2X_{n1})]'$.[23] The second GMM estimator GMM2 is based on the moment vector $[V_n'(G_n - \mathrm{tr}(G_n)I_n/n)V_n, V_n'(X_n, G_nX_n\beta_0)]'$. The second moment vector is the best one in the case of normal disturbances (Lee, 2007), in the sense that the resulting GMM estimator has the same asymptotic distribution as the ML estimator. The unknown parameters in the moment vector for GMM2 are replaced with initial estimates which correspond to GMM1 estimates.

(1.2) In the heteroskedastic case, following Liu and Yang (2015), the score vector of the QML is modified to obtain a consistent estimator, which is denoted as MQML. We consider two robust GMM estimators RGMM1 and RGMM2, which are based on, respectively, $[V_n'W_nV_n, V_n'(W_n^2 - \mathrm{diag}(W_n^2))V_n, V_n'(X_n, W_nX_{n1}, W_n^2X_{n1})]'$ and $[V_n'(G_n - \mathrm{diag}(G_n))V_n, V_n'(X_n, G_nX_n\beta_0)]'$, where the unknown parameters in the latter are replaced with robust GMM estimates in RGMM1.

(2) For the estimation of the SE model:

(2.1) In the homoskedastic case, we consider the QML, LS and also two GMM estimators. For the LS, with the LS estimator $\hat{\beta}_n$ of β, the residual vector $\hat{V}_n = Y_n - X_n\hat{\beta}_n$ is computed and the spatial error ρ is estimated with the moment vector $[\hat{V}_n'M_n\hat{V}_n, \hat{V}_n'(M_n^2 - \mathrm{tr}(M_n^2)I_n/n)\hat{V}_n]$. The first GMM estimator GMM1 is based on the moment vector $[V_n'M_nV_n, V_n'(M_n^2 - \mathrm{tr}(M_n^2)I_n/n)V_n, V_n'(X_n, M_nX_{n1}, M_n^2X_{n1})]'$, and the second GMM estimator GMM2 is based on the moment vector $[V_n'(H_n - \mathrm{tr}(H_n)I_n/n)V_n, V_n'R_nX_n]'$, which is the best one in the case of normal disturbances (Liu et al., 2010).

(2.2) In the heteroskedastic case, the modified QML estimator MQML is also considered. For the LS, the spatial error ρ is estimated with the moment vector $[\hat{V}_n'M_n\hat{V}_n, \hat{V}_n'(M_n^2 - \mathrm{diag}(M_n^2))\hat{V}_n]$. The two robust GMM estimators RGMM1 and RGMM2 are based, respectively, on the moments

$$[V_n'M_nV_n, V_n'(M_n^2 - \mathrm{diag}(M_n^2))V_n, V_n'(X_n, M_nX_{n1}, M_n^2X_{n1})]'$$

and $[V_n'(H_n - \mathrm{diag}(H_n))V_n, V_n'R_nX_n]'$. The unknown parameters in R_n and H_n of the moment vectors for GMM2 and RGMM2 are replaced, respectively, with the GMM1 and RGMM1 estimates.

[22] As W_n is row-normalized, $W_nl_n = l_n$ in W_nX_n and $W_n^2l_n = l_n$ in $W_n^2X_n$. So the constant term is excluded in X_{n1} to avoid multicollinearity.

[23] The matrices $(I_n - \lambda W_n)$ and $(I_n - \rho M_n)$ are singular at $\lambda = \rho = 1$ since W_n and M_n are row-normalized. In all kinds of test statistics, $(I_n - \lambda W_n)^{-1}$ and $(I_n - \rho M_n)^{-1}$ need to be computed at the estimates of λ and ρ. When the estimates are close to 1, tests are expected not to perform well. Thus, we append asymptotically ignorable terms $-\frac{1}{n}\log(1-\lambda)$ and $-\frac{1}{n}\log(1-\rho)$ to relevant GMM criterion functions, which bound the estimates of λ and ρ away from 1 but without distortion on the asymptotic distribution of the GMM estimator. For the QML, by using characteristic polynomials of $|I_n - \lambda W_n|$ and $|I_n - \rho M_n|$, similar terms $\log(1-\lambda)$ and $\log(1-\rho)$ also appear in log QML functions. Test statistics computed with the adjusted GMM estimates are observed to perform better than those without adjustments in our Monte Carlo experiments.

With these estimates, for the test of $\rho_0 = 0$, we have the score-based OPG test statistics S_{QML}, S_{2SLS}, S_{GMM1} and S_{GMM2} in the homoskedastic case, and S_{MQML}, S_{2SLS}, S_{RGMM1} and S_{RGMM2} in the heteroskedastic case. For the test of $\lambda_0 = 0$, instead of S_{2SLS}, we have the score-based OPG test S_{LS} with the LS estimate, and other score-based tests are similar in notations. As mentioned in Section 2, S_{QML} is the LM test with OPG variance estimate.

For the gradient-based OPG tests in the GMM framework, two moment vectors are considered for the construction of those GMM test statistics for either $\rho_0 = 0$ or $\lambda_0 = 0$.

(1) For the test of $\rho_0 = 0$:

(1.1) In the homoskedastic case, the first test G_{GMM1} is based on the moment vector $[V_n' W_n V_n, V_n'(W_n^2 - \text{tr}(W_n^2) I_n/n) V_n, V_n'(X_n, W_n X_{n2}, W_n^2 X_{n2})]'$ with GMM1 estimates for the SAR model. The second test G_{GMM2} for $\rho_0 = 0$ is based on the moment vector $[V_n'(G_n - \text{tr}(G_n) I_n/n) V_n, V_n' M_n V_n, V_n'(X_n, G_n X_n \beta_0)]$, which is the best moment vector for the SARAR model with homoskedastic disturbances evaluated at $\rho_0 = 0$, where the unknown parameters β_0 and λ_0 in G_n are replaced with the GMM2 estimates for the SAR model.

(1.2) In the heteroskedastic case, the first test G_{RGMM1} is based on the moment vector $[V_n' W_n V_n, V_n'(W_n^2 - \text{diag}(W_n^2)) V_n, V_n'(X_n, W_n X_{n2}, W_n^2 X_{n2})]'$ and the robust RGMM1 estimate, and the second test G_{RGMM2} is based on the moment vector $[V_n'(G_n - \text{diag}(G_n)) V_n, V_n' M_n V_n, V_n'(X_n, G_n X_n \beta_0)]$ and the robust RGMM2 estimate.

(2) For the test of $\lambda_0 = 0$:

(2.1) In the homoskedastic case, the first test G_{GMM1} is based on the moment vector $[V_n' M_n V_n, V_n'(M_n^2 - \text{tr}(M_n^2) I_n/n) V_n, V_n'(X_n, M_n X_{n2}, M_n^2 X_{n2})]'$ and the GMM1 estimate, and the second test G_{GMM2} is based on the moment vector $[V_n'(\ddot{W}_n - \text{tr}(\ddot{W}_n) I_n/n) V_n, V_n'(H_n - \text{tr}(H_n) I_n/n) V_n, V_n'(R_n X_n, R_n W_n X_n \beta_0)]$, which is the best moment vector for the SARAR model with homoskedastic disturbances evaluated at $\lambda_0 = 0$.

(2.2) In the heteroskedastic case, the first test G_{RGMM1} is based on the moment vector $[V_n' M_n V_n, V_n'(M_n^2 - \text{diag}(M_n^2)) V_n, V_n'(X_n, M_n X_{n2}, M_n^2 X_{n2})]'$ and the robust RGMM1 estimate, and the second test G_{RGMM2} is based on the moment vector $[V_n'(\ddot{W}_n - \text{diag}(\ddot{W}_n)) V_n, V_n'(H_n - \text{diag}(H_n)) V_n, V_n'(R_n X_n, R_n W_n X_n \beta_0)]$ and the robust RGMM2 estimate.

To compare OPG tests with other classical tests, we also compute some Wald tests and GMM distance difference tests.[24] Two GMM distance difference tests are considered for $\rho_0 = 0$ and $\lambda_0 = 0$.

(1) For the test of $\rho_0 = 0$:

[24] Recall that the score-based OPG test with the QML estimate is a properly extended LM test, and the gradient-based OPG tests with a restricted optimal GMM estimate is the GMM gradient test. When the disturbances are non-normal or heteroskedastic, we only have a quasi likelihood function but not a likelihood function, so the LR statistic does not follow an asymptotically chi-squared distribution. Thus the LR statistic is not computed.

(1.1) In the homoskedastic case, both the SAR model and SARAR model are estimated with the moment vector $[V_n'W_nV_n, V_n'(W_n^2-\text{tr}(W_n^2)I_n/n)V_n, V_n'(X_n, W_nX_{n2}, W_n^2X_{n2})]'$ to compute the first distance difference test D_{GMM1}, and both models are estimated with the moment vector $[V_n'(G_n-\text{tr}(G_n)I_n/n)V_n, V_n'M_nV_n, V_n'(X_n, G_nX_n\beta_0)]$ to compute the second test D_{GMM2}, where the unknown parameters β_0 and λ_0 in G_n are replaced with the GMM2 estimates for the SAR model.

(1.2) In the heteroskedastic case, the first distance difference test D_{RGMM1} is based on the moment vector $[V_n'W_nV_n, V_n'(W_n^2 - \text{diag}(W_n^2))V_n, V_n'(X_n, W_nX_{n2}, W_n^2X_{n2})]'$, and the second one D_{RGMM2} is based on $[V_n'(G_n - \text{diag}(G_n))V_n, V_n'M_nV_n, V_n'(X_n, G_nX_n\beta_0)]$ and the robust estimate RGMM2 for the SAR model.

(2) For the test of $\lambda_0 = 0$:

(2.1) In the homoskedastic case, both the SE model and SARAR model are estimated with the moment vector $[V_n'M_nV_n, V_n'(M_n^2 - \text{tr}(M_n^2)I_n/n)V_n, V_n'(X_n, M_nX_{n2}, M_n^2X_{n2})]'$ to compute the first distance difference test D_{GMM1}, and both models are estimated with the moment vector $[V_n'(\ddot{W}_n - \text{tr}(\ddot{W}_n)I_n/n)V_n, V_n'(H_n - \text{tr}(H_n)I_n/n)V_n, V_n'(R_nX_n, R_nW_nX_n\beta_0)]$ to compute the second test D_{GMM2}, where the unknown parameters β_0 and ρ_0 in H_n are replaced with the GMM2 estimates for the SE model.

(2.2) In the heteroskedastic case, the first distance difference test D_{RGMM1} is based on the moment vector $[V_n'M_nV_n, V_n'(M_n^2 - \text{diag}(M_n^2))V_n, V_n'(X_n, M_nX_{n2}, M_n^2X_{n2})]'$, and the second one D_{RGMM2} is based on $[V_n'(\ddot{W}_n - \text{diag}(\ddot{W}_n))V_n, V_n'(H_n - \text{diag}(H_n))V_n, V_n'(R_nX_n, R_nW_nX_n\beta_0)]$ and the robust RGMM2 estimate for the SE model.

Wald tests in the GMM framework are computed using the two GMM estimates for the SARAR model in each case as described above. They are denoted by W_{GMM1} and W_{GMM2} in the homoskedastic case, and denoted by W_{RGMM1} and W_{RGMM2} in the heteroskedastic case. The Wald test in the likelihood framework is denoted by W_{QML} in the homoskedastic case, and denote by W_{MQML} in the heteroskedastic case.

Tables 1–4 report empirical sizes of various tests under the nominal size of 5%. The tests generally have small size distortions except S_{2SLS} that tests $\rho_0 = 0$ and Wald tests. From Table 1 on testing $\rho_0 = 0$, the size of S_{2SLS} can be relatively higher when R^2 is small and λ_0 is large for some cases. This can be due to relatively poor performance of the 2SLS estimate, since stronger spatial lag dependence and less variation in $X_n\beta_0$ relative to that in disturbances imply that those IVs are less relevant and the 2SLS estimators have large variances, and some estimated values of λ might be too large for the SAR process to be stable. Wald tests have large size distortions in most cases for $n = 144$ and the distortions are more severe in the heteroskedastic case. With the larger sample size $n = 400$, empirical sizes of Wald test are closer to 5%. Tests other than S_{2SLS} and Wald tests typically have size distortions smaller than 1 percentage points and occasionally slightly larger than 2 percentage points. In particular, from Tables 3 and 4 on tests of $\lambda_0 = 0$, even score-based OPG tests with LS estimates have relatively small size distortions.

[Tables 1–4 about here.]

Powers of the tests are presented in Tables 5–12. We first focus on Table 5 for tests of $\rho_0 = 0$ with normal homoskedastic disturbances in V_n. Since S_{2SLS} and Wald tests can have large size distortions, we only investigate powers of other tests. Some general orders of the tests in terms of power are: $S_{QML} \approx D_{GMM2} > S_{GMM2} > S_{GMM1}$, $D_{GMM2} > G_{GMM2} > G_{GMM1}$, and $D_{GMM2} > D_{GMM1} > G_{GMM1}$. S_{GMM1} is more powerful than G_{GMM2} when $R^2 = 0.8$ but less powerful when $R^2 = 0.4$. The gap between the powers of S_{QML} and S_{GMM2} is generally small. Tests are more powerful as sample sizes increase, as ρ_0 in the DGP increases, as R^2 increases from 0.4 to 0.8, and as the spatial weights matrices becomes sparser (from the queen matrix to the rook matrix). From Tables 6–8, similar patterns are observed when the disturbances are chi-squared distributed and/or heteroskedastic.

Tables 9–12 report powers of tests for $\lambda_0 = 0$. Orders of powers are similar to those for $\rho_0 = 0$, other than no clear pattern for the comparison of S_{GMM1} and G_{GMM2}. The powers of S_{GMM2} and S_{QML} are almost the same except for some cases with $\rho_0 = 0.8$ and large λ_0. S_{LS} has much lower power than other tests for some cases with $\rho_0 = 0.8$ and/or $\lambda_0 = 0.8$. We still observe larger power for larger sample sizes, larger deviations from the null hypothesis (i.e., larger λ_0 in the DGP), larger R^2 and sparser spatial weights matrices.

[Tables 5–12 about here.]

5 Conclusion

In this paper, we propose to use the formula of $C(\alpha)$-type statistics to derive systematically valid OPG variants of test statistics for SARAR models with homoskedastic or heteroskedastic disturbances. In particular, we derive score-based and gradient-based OPG tests for SARAR models with homoskedastic disturbances, and robust OPG tests for models with unknown heteroskedastic disturbances. Those OPG statistics are also distribution free. As variances of quadratic moments in an SARAR model with homoskedastic disturbances involve third and fourth higher order moment parameters under non-normality, the OPG formulation of test statistics can avoid their estimation. The formulation of those OPG tests for SAR models is thus computationally simpler. In our Monte Carlo experiments, OPG tests with QML and GMM estimates have small size distortion and are generally powerful.

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Appendix A Some expressions

For score-based OPG tests under homoskedasticity of Section 2.1, at a general θ_0,

$$-\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\lambda^2}\big) = \frac{1}{\sigma_0^2}(R_nG_nX_n\beta_0)'R_nG_nX_n\beta_0 + \mathrm{tr}(\ddot{G}_n^{(s)}\ddot{G}_n), \quad -\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\lambda\partial\rho}\big) = \mathrm{tr}(H_n^{(s)}\ddot{G}_n), \tag{22}$$

$$-\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\lambda\partial\beta}\big) = \frac{1}{\sigma_0^2}X_n'R_n'R_nG_nX_n\beta_0, \quad -\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\lambda\partial\sigma^2}\big) = \frac{1}{\sigma_0^2}\mathrm{tr}(G_n), \tag{23}$$

$$-\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\rho^2}\big) = \mathrm{tr}(H_n^{(s)}H_n), \quad -\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\rho\partial\beta}\big) = 0, \tag{24}$$

$$-\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\rho\partial\sigma^2}\big) = \frac{1}{\sigma_0^2}\mathrm{tr}(H_n), \quad -\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\beta\partial\beta'}\big) = \frac{1}{\sigma_0^2}X_n'R_n'R_nX_n, \tag{25}$$

$$-\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\beta\partial\sigma^2}\big) = 0, \quad -\mathrm{E}\big(\frac{\partial^2 \ln L_n(\theta_0)}{\partial\sigma^4}\big) = \frac{n}{2\sigma_0^4}. \tag{26}$$

For score-based OPG tests under unknown heteroskedasticity of Section 2.2, at a general δ_0,

$$-\mathrm{E}\big(\frac{\partial g_{n\lambda}(\delta_0)}{\partial\lambda}\big) = \mathrm{tr}[(\ddot{G}_n - \mathrm{diag}(\ddot{G}_n))^{(s)}\ddot{G}_n\Sigma_n] + (R_nG_nX_n\beta_0)'R_nG_nX_n\beta_0, \tag{27}$$

$$-\mathrm{E}\big(\frac{\partial g_{n\lambda}(\delta_0)}{\partial\rho}\big) = -\mathrm{E}\big(\frac{\partial g_{n\rho}(\delta_0)}{\partial\lambda}\big) = \mathrm{tr}[(H_n - \mathrm{diag}(H_n))^{(s)}\ddot{G}_n\Sigma_n], \tag{28}$$

$$-\mathrm{E}\big(\frac{\partial g_{n\lambda}(\delta_0)}{\partial\beta}\big) = -\mathrm{E}\big(\frac{\partial g_{n\beta}(\delta_0)}{\partial\lambda}\big) = X_n'R_n'R_nG_nX_n\beta_0, \tag{29}$$

$$-\mathrm{E}\big(\frac{\partial g_{n\rho}(\delta_0)}{\partial\rho}\big) = \mathrm{tr}[(H_n - \mathrm{diag}(H_n))^{(s)}H_n\Sigma_n], \tag{30}$$

$$-\mathrm{E}\big(\frac{\partial g_{n\rho}(\delta_0)}{\partial\beta}\big) = -\mathrm{E}\big(\frac{\partial g_{n\beta}(\delta_0)}{\partial\rho}\big) = 0, \tag{31}$$

$$-\mathrm{E}\big(\frac{\partial g_{n\beta}(\delta_0)}{\partial\beta'}\big) = X_n'R_n'R_nX_n, \tag{32}$$

For gradient-based OPG tests under homoskedasticity of Section 3.1, at a general δ_0,

$$\mathrm{E}\big(\frac{\partial g_n(\delta_0)}{\partial\lambda}\big) = -\begin{pmatrix} \sigma_0^2\,\mathrm{tr}(P_{n1}^{(s)}\ddot{G}_n) \\ \vdots \\ \sigma_0^2\,\mathrm{tr}(P_{n,k_p}^{(s)}\ddot{G}_n) \\ F_n'R_nG_nX_n\beta_0 \end{pmatrix}, \quad \mathrm{E}\big(\frac{\partial g_n(\delta_0)}{\partial\rho}\big) = -\begin{pmatrix} \sigma_0^2\,\mathrm{tr}(P_{n1}^{(s)}H_n) \\ \vdots \\ \sigma_0^2\,\mathrm{tr}(P_{n,k_p}^{(s)}H_n) \\ 0_{k_f\times 1} \end{pmatrix}, \quad \text{and } \mathrm{E}\big(\frac{\partial g_n(\delta_0)}{\partial\beta'}\big) = -\begin{pmatrix} 0_{k_p\times k} \\ F_n'R_nX_n \end{pmatrix}. \tag{33}$$

For gradient-based OPG tests under unknown heteroskedasticity of Section 3.2, at a general δ_0,

$$\mathrm{E}\big(\frac{\partial g_n(\delta_0)}{\partial\lambda}\big) = -\begin{pmatrix} \mathrm{tr}(P_{n1}^{(s)}\ddot{G}_n\Sigma_n) \\ \vdots \\ \mathrm{tr}(P_{n,k_p}^{(s)}\ddot{G}_n\Sigma_n) \\ F_n'R_nG_nX_n\beta_0 \end{pmatrix}, \quad \mathrm{E}\big(\frac{\partial g_n(\delta_0)}{\partial\rho}\big) = -\begin{pmatrix} \mathrm{tr}(P_{n1}^{(s)}H_n\Sigma_n) \\ \vdots \\ \mathrm{tr}(P_{n,k_p}^{(s)}H_n\Sigma_n) \\ 0_{k_f\times 1} \end{pmatrix}, \quad \text{and } \mathrm{E}\big(\frac{\partial g_n(\delta_0)}{\partial\beta'}\big) = -\begin{pmatrix} 0_{k_p\times k} \\ F_n'R_nX_n \end{pmatrix}. \tag{34}$$

Appendix B Lemmas

Lemma 1. *Let* $P_{nl}(\theta) = [p_{nl,ij}(\theta)]$ *be* $n \times n$ *nonstochastic matrices which are bounded in row and column sum norms uniformly in* $\theta \in \Theta$*, for* $l = 1, \ldots, s$*. If* $\sup_n \sup_{1 \le j \le n} \mathrm{E}\,|v_{nj}|^s < \infty$*, then* $\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n \prod_{l=1}^s \sum_{j=1}^n p_{nl,ij}(\theta) v_{nj}| = O_p(1)$*.*

Proof. For $s = 1$,

$$\sup_{\theta \in \Theta} \Big| \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n p_{n1,ij}(\theta) v_{nj} \Big| \le \frac{1}{n} \sum_{j=1}^n |v_{nj}| \sup_{\theta \in \Theta} \sum_{i=1}^n |p_{n1,ij}(\theta)| \le \frac{1}{n} \sum_{j=1}^n |v_{nj}| \sup_{\theta \in \Theta} \|P_{n1}(\theta)\|_1 = O_p(1),$$

by Markov's inequality. For $s > 1$, there exists a finite r such that $\frac{1}{r} + \frac{1}{s} = 1$. Hölder's inequality implies that

$$\begin{aligned}
\sum_{j=1}^n |p_{nl,ij}(\theta) v_{nj}| &= \sum_{j=1}^n |p_{nl,ij}(\theta)|^{\frac{1}{r}} |p_{nl,ij}(\theta)|^{\frac{1}{s}} |v_{nj}| \le \Big[\sum_{j=1}^n (|p_{nl,ij}(\theta)|^{\frac{1}{r}})^r \Big]^{\frac{1}{r}} \Big[\sum_{j=1}^n (|p_{nl,ij}(\theta)|^{\frac{1}{s}} |v_{nj}|)^s \Big]^{\frac{1}{s}} \\
&\le c^{\frac{1}{r}} \Big[\sum_{j=1}^n |p_{nl,ij}(\theta)| |v_{nj}|^s \Big]^{\frac{1}{s}} \le c^{\frac{1}{r}} \Big[\sum_{j=1}^n \Big(\sum_{l=1}^s |p_{nl,ij}(\theta)| \Big) |v_{nj}|^s \Big]^{\frac{1}{s}},
\end{aligned}$$

where $c = \sup_{l=1,\cdots,s} \sup_{\theta \in \Theta} \|P_{nl}(\theta)\|_\infty$. It follows that

$$\Big| \frac{1}{n} \sum_{i=1}^n \prod_{l=1}^s \sum_{j=1}^n p_{nl,ij}(\theta) v_{nj} \Big| \le \frac{1}{n} c^{s/r} \sum_{i=1}^n \Big[\sum_{j=1}^n \Big(\sum_{l=1}^s |p_{nl,ij}(\theta)| \Big) |v_{nj}|^s \Big].$$

Thus the conclusion follows by a similar argument as before (as if $s = 1$) with $\frac{1}{n} \sum_{j=1}^n |v_{nj}|^s = O_p(1)$ and a matrix with (i, j)th element equal to $\sum_{l=1}^s |p_{nl,ij}(\theta)|$. □

The following lemma shows the consistency of the OPG estimator for the covariance of two linear-quadratic forms of V_n in the SARAR model (1). Let $V_n(\delta) = [v_{ni}(\delta)] = R_n(\rho)[S_n(\lambda) Y_n - X_n \beta]$, where $\delta = (\lambda, \rho, \beta')'$, and $\theta = (\delta', \sigma^2)'$ is of dimension $k + 3$.

Lemma 2. *Suppose that* $A_{n1}(\delta) = [a_{n1,ij}(\delta)]$ *and* $A_{n2}(\delta) = [a_{n2,ij}(\delta)]$ *are* $n \times n$ *nonstochastic symmetric matrices,* $b_{n1}(\delta) = [b_{n1,i}(\delta)]$ *and* $b_{n2}(\delta) = [b_{n2,i}(\delta)]$ *are* $n \times 1$ *nonstochastic vectors, and their elements are functions of* $\delta \in \Delta$*. Assume that each element of* $A_{n1}(\delta)$*,* $A_{n2}(\delta)$*,* $b_{n1}(\delta)$ *and* $b_{n2}(\delta)$ *is differentiable with respect to* δ*,* $A_{n1}(\delta)$*,* $A_{n2}(\delta)$*,* $\frac{\partial A_{n1}(\delta)}{\partial \delta_j}$ *and* $\frac{\partial A_{n2}(\delta)}{\partial \delta_j}$ *for* $j = 1, \ldots, k+2$ *are bounded in both row and column sum norms uniformly on* Δ*, and* $b_{n1}(\delta)$*,* $b_{n2}(\delta)$*,* $\frac{\partial b_{n1}(\delta)}{\partial \delta_j}$ *and* $\frac{\partial b_{n2}(\delta)}{\partial \delta_j}$ *for* $j = 1, \ldots, k+3$ *are bounded in row sum norm uniformly on* Δ*.*

Let $\xi_{nr,i}(\theta) = a_{nr,ii}(\delta)[v_{ni}^2(\delta) - \sigma^2] + 2 v_{ni}(\delta) \sum_{j=1}^{i-1} a_{nr,ij}(\delta) v_{nj}(\delta) + b_{nr,i}(\delta) v_{ni}(\delta)$ *for* $r = 1, 2$ *if the disturbances* v_{ni}*'s are homoskedastic, and* $\xi_{nr,i}(\delta) = 2 v_{ni}(\delta) \sum_{j=1}^{i-1} a_{nr,ij}(\delta) v_{nj}(\delta) + b_{nr,i}(\delta) v_{ni}(\delta)$ *for* $r = 1, 2$ *if the disturbances* v_{ni}*'s are heteroskedastic. Then,*

(i) under Assumptions 1 and 2, if $\hat{\theta}_n = \theta_0 + o_p(1)$*,* $\frac{1}{n} \sum_{i=1}^n \xi_{n1,i}(\hat{\theta}_n) \xi_{n2,i}(\hat{\theta}_n) = \frac{1}{n} \sum_{i=1}^n \mathrm{E}[\xi_{n1,i}(\theta_0) \xi_{n2,i}(\theta_0)] + o_p(1)$*, and*

(ii) under Assumptions 1 and 4 with $\mathrm{diag}(A_{n1}(\delta)) = \mathrm{diag}(A_{n2}(\delta)) = 0$*, if* $\hat{\delta}_n = \delta_0 + o_p(1)$*,* $\frac{1}{n} \sum_{i=1}^n \xi_{n1,i}(\hat{\delta}_n) \xi_{n2,i}(\hat{\delta}_n) = \frac{1}{n} \sum_{i=1}^n \mathrm{E}[\xi_{n1,i}(\delta_0) \xi_{n2,i}(\delta_0)] + o_p(1)$*.*

Proof. (i) By the mean value theorem,

$$\frac{1}{n}\sum_{i=1}^{n}\xi_{n1,i}(\hat{\theta}_n)\xi_{n2,i}(\hat{\theta}_n)=\frac{1}{n}\sum_{i=1}^{n}\xi_{n1,i}(\theta_0)\xi_{n2,i}(\theta_0)+\sum_{l=1}^{k+3}\frac{1}{n}\sum_{i=1}^{n}\Big[\frac{\partial\xi_{n1,i}(\check{\theta}_n)}{\partial\theta_l}\xi_{n2,i}(\check{\theta}_n)+\xi_{n1,i}(\check{\theta}_n)\frac{\partial\xi_{n2,i}(\check{\theta}_n)}{\partial\theta_l}\Big](\hat{\theta}_{nl}-\theta_{0l}),\tag{35}$$

where $\check{\theta}_n$ lies between θ_0 and $\hat{\theta}_n$. We shall show that the second term on the r.h.s. of the above equation goes to zero in probability. Note that $\xi_{nr,i}(\theta)=a_{nr,ii}(\delta)[(e_{ni}'V_n(\delta))^2-\sigma^2]+2e_{ni}'V_n(\delta)e_{ni}'\text{tril}[A_n(\delta)]V_n(\delta)+b_{nr,i}(\delta)e_{ni}'V_n(\delta)$, where e_{ni} is the ith column of I_n. Under the assumptions in the lemma, $\text{tril}[A_n(\delta)]$ and $\frac{\partial \text{tril}[A_n(\delta)]}{\partial\delta_l}$ for $l=1,\ldots,k+2$ are bounded in both row and column sum norms uniformly on Δ. Since $Y_n=S_n^{-1}X_n\beta_0+S_n^{-1}R_n^{-1}V_n$,

$$\begin{aligned}V_n(\delta)&=[R_n+(\rho_0-\rho)M_n]\{[S_n+(\lambda_0-\lambda)W_n](S_n^{-1}X_n\beta_0+S_n^{-1}R_n^{-1}V_n)-X_n\beta\}\\&=R_nX_n(\beta_0-\beta)+(\lambda_0-\lambda)R_nG_nX_n\beta_0+M_nX_n(\beta_0-\beta)(\rho_0-\rho)+(\lambda_0-\lambda)(\rho_0-\rho)M_nG_nX_n\beta_0\\&\quad+(\lambda_0-\lambda)\ddot{G}_nV_n+(\rho_0-\rho)H_nV_n+(\rho_0-\rho)(\lambda_0-\lambda)M_nG_nR_n^{-1}V_n+V_n,\end{aligned}\tag{36}$$

which is linear in V_n and quadratic in δ. In (36), for given δ, terms that do not involve V_n have uniformly bounded elements, and terms that involve V_n have matrices bounded in both row and column sum norms in front of V_n. Then we can use (36) to expand $\frac{1}{n}\sum_{i=1}^{n}[\frac{\partial\xi_{n1,i}(\theta)}{\partial\theta_l}\xi_{n2,i}(\theta)+\xi_{n1,i}(\theta)\frac{\partial\xi_{n2,i}(\theta)}{\partial\theta_l}]$ as a polynomial of the elements of $(\theta-\theta_0)$, whose coefficients have the form of the term in Lemma 1, by which $\frac{1}{n}\sum_{i=1}^{n}[\frac{\partial\xi_{n1,i}(\theta)}{\partial\theta_l}\xi_{n2,i}(\theta)+\xi_{n1,i}(\theta)\frac{\partial\xi_{n2,i}(\theta)}{\partial\theta_l}]=O_p(1)$ uniformly in a neighborhood of θ_0. Since $\hat{\theta}_n=\theta_0+o_p(1)$, the second term on the r.h.s. of (35) is $o_p(1)$. Thus,

$$\frac{1}{n}\sum_{i=1}^{n}\xi_{n1,i}(\hat{\theta}_n)\xi_{n2,i}(\hat{\theta}_n)=\frac{1}{n}\sum_{i=1}^{n}\xi_{n1,i}(\theta_0)\xi_{n2,i}(\theta_0)+o_p(1).$$

In the proof of Theorem 1 in Baltagi and Yang (2013), it is shown that $\frac{1}{n}\sum_{i=1}^{n}\xi_{n1,i}^2(\theta_0)-\frac{1}{n}\sum_{i=1}^{n}\text{E}[\xi_{n1,i}^2(\theta_0)]=o_p(1)$. This is proved by rewriting it as a sum of several terms, where each term is a sum of martingale differences, so a law of large numbers for martingale difference arrays (Davidson, 1994, p. 299) can be applied to each term. As $\xi_{n1,i}(\theta_0)$ is quadratic in v_{nj}'s, it is enough to require the existence of finite moments of v_{nj}'s of order slightly higher than the fourth, i.e., $\sup_n\sup_{1\le j\le n}\text{E}\,|v_{nj}|^{4+\alpha}<\infty$ for some $\alpha>0$. We can similarly show that $\frac{1}{n}\sum_{i=1}^{n}\xi_{n1,i}(\theta_0)\xi_{n2,i}(\theta_0)-\frac{1}{n}\sum_{i=1}^{n}\text{E}[\xi_{n1,i}(\theta_0)\xi_{n2,i}(\theta_0)]+o_p(1)$. Hence, (i) in the lemma follows. (ii) is similar. □

Appendix C Proofs

Proof of Proposition 1. By the mean value theorem,

$$[I_{k_1},-\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\frac{1}{\sqrt{n}}\frac{\partial\ln L_n(0,\hat{\delta}_{2n})}{\partial\delta}=[I_{k_1},-\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\Big[\frac{1}{\sqrt{n}}\frac{\partial\ln L_n(0,\delta_{20})}{\partial\delta}+\frac{1}{n}\frac{\partial^2\ln L_n(0,\check{\delta}_{2n})}{\partial\delta\partial\delta_2'}\sqrt{n}(\hat{\delta}_{2n}-\delta_{20})\Big],$$

where $\check{\delta}_{2n}$ can vary across rows of the above equation, but each $\check{\delta}_{2n}$ lies between $\hat{\delta}_{2n}$ and δ_{20}. It is proved, for example, in Proposition 4 in Jin and Lee (2013) that $\frac{1}{n}\frac{\partial^2\ln L_n(\hat{\delta}_n)}{\partial\delta\partial\delta'}=\frac{1}{n}\text{E}(\frac{\partial^2 L_n(\delta_0)}{\partial\delta\partial\delta'})+o_p(1)$ for any consistent estimator $\hat{\delta}_n$, where $\lim_{n\to\infty}\frac{1}{n}\text{E}(\frac{\partial^2 L_n(\delta_0)}{\partial\delta\partial\delta'})$ is nonsingular under Assumption 2(*ii*). Also, $\frac{1}{n}\hat{\Omega}_{n,12}=\frac{1}{n}\Omega_{n,12}+o_p(1)$

and $\frac{1}{n}\hat{\Omega}_{n,22} = \frac{1}{n}\Omega_{n,22} + o_p(1)$ by Lemma 12 in Jin and Lee (2013). Thus,

$$[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\frac{1}{\sqrt{n}}\frac{\partial L_n(0,\hat{\delta}_{2n})}{\partial \delta} = [I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\Big[\frac{1}{\sqrt{n}}\frac{\partial L_n(0,\delta_{20})}{\partial \delta} + \begin{pmatrix}\frac{1}{n}\Omega_{n,12}\\ \frac{1}{n}\Omega_{n,22}\end{pmatrix}\sqrt{n}(\hat{\delta}_{2n}-\delta_{20})\Big] + o_p(1)$$
$$= [I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\frac{1}{\sqrt{n}}\frac{\partial L_n(0,\delta_{20})}{\partial \delta} + o_p(1).$$

Hence, the result in the proposition holds. □

Proof of Proposition 2. By the central limit theorem for linear-quadratic forms in Kelejian and Prucha (2001, p. 227, Theorem 1),

$$\frac{1}{\sqrt{n}}\Upsilon_n = [I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\frac{1}{\sqrt{n}}\frac{\partial L_n(0,\delta_{20})}{\partial \delta}$$
$$\xrightarrow{d} N\big(0, \lim_{n\to\infty}[I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\frac{1}{n}\,\mathrm{E}\big(\frac{\partial L_n(0,\delta_{20})}{\partial \delta}\frac{\partial L_n(0,\delta_{20})}{\partial \delta'}\big)[I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]'\big).$$

As $\lim_{n\to\infty}\frac{1}{n}\,\mathrm{E}\big(\frac{\partial L_n(0,\delta_{20})}{\partial \delta}\frac{\partial L_n(0,\delta_{20})}{\partial \delta'}\big)$ is assumed to be nonsingular in Assumption 2(*iii*), the limiting covariance matrix of $\frac{1}{\sqrt{n}}\Upsilon_n$ is nonsingular, and its rank is equal k_1. By Lemma A.3 in Lee (2004b), $S_n^{-1}(\lambda)$ and its derivative are bounded in both row and column sum norms uniformly in a neighborhood of λ_0. Similar results hold for $R_n^{-1}(\rho)$. As $\hat{\delta}_{2n} = \delta_{20} + o_p(1)$, by Lemma 2, $\frac{1}{n}\hat{\varphi}_n'\hat{\varphi}_n = \frac{1}{n}\sum_{i=1}^n \hat{\xi}_{ni}\hat{\xi}_{ni}' = \frac{1}{n}\sum_{i=1}^n \mathrm{E}(\xi_{ni}\xi_{ni}') + o_p(1)$, where $\frac{1}{n}\sum_{i=1}^n \mathrm{E}(\xi_{ni}\xi_{ni}') = \frac{1}{n}[I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]\,\mathrm{E}(\frac{\partial \ln L_n(0,\delta_{20})}{\partial \delta}\frac{\partial \ln L_n(0,\delta_{20})}{\partial \delta'})[I_{k_1}, -\Omega_{n,12}\Omega_{n,22}^{-1}]'$. Thus, by Proposition 1, the result in the proposition holds. □

Proof of Proposition 3. The proof is similar to that for Proposition 1 except for $\frac{1}{n}\hat{\Omega}_{n,12} = \frac{1}{n}\Omega_{n,12} + o_p(1)$ and $\frac{1}{n}\hat{\Omega}_{n,12} = \frac{1}{n}\Omega_{n,12} + o_p(1)$, where the specific expressions of $\Omega_{n,12}$ and $\Omega_{n,22}$, and hence, $\hat{\Omega}_{n,12}$ and $\hat{\Omega}_{n,22}$, are in (27)–(32). By the consistency of $\hat{\delta}_n$, we need to show that (1) $\frac{1}{n}X_n'G_n'(\hat{\lambda}_n)R_n'(\hat{\rho}_n)R_n(\hat{\rho}_n)G_n(\hat{\lambda}_n)X_n = \frac{1}{n}X_n'G_n'R_n'R_nG_nX_n + o_p(1)$, (2) $\frac{1}{n}X_n'R_n'(\hat{\rho}_n)R_n(\hat{\rho}_n)G_n(\hat{\lambda}_n)X_n = \frac{1}{n}X_n'R_n'R_nG_nX_n + o_p(1)$, (3) $\frac{1}{n}X_n'R_n'(\hat{\rho}_n)R_n(\hat{\rho}_n)X_n = \frac{1}{n}X_n'R_n'R_nX_n + o_p(1)$, and (4) $\frac{1}{n}\operatorname{tr}[A_n(\hat{\phi}_n)\hat{\Sigma}_n] = \frac{1}{n}\operatorname{tr}[A_n(\phi_0)\Sigma_n] + o_p(1)$, where $A_n(\phi) = [\ddot{G}_n^s(\phi) - \operatorname{diag}(\ddot{G}_n^s(\phi))]\ddot{G}_n(\phi)$, $[\ddot{G}_n^s(\phi) - \operatorname{diag}(\ddot{G}_n^s(\phi))]H_n(\rho)$, $[H_n^s(\rho) - \operatorname{diag}(H_n^s(\rho))]\ddot{G}_n(\phi)$ or $[H_n^s(\rho) - \operatorname{diag}(H_n^s(\rho))]H_n(\rho)$. Since $R_n(\rho)$ is linear in ρ and $G_n(\lambda)$ is bounded in both row and column sum norms in a neighborhood of λ_0 by Lemma A.3 in Lee (2004b), (1)–(3) follow by asymptotic expansions via the mean value theorem. For (4), we shall show that (i) $\frac{1}{n}\operatorname{tr}[A_n(\hat{\phi}_n)\hat{\Sigma}_n] = \frac{1}{n}\operatorname{tr}(A_n\hat{\Sigma}_n) + o_p(1)$, (ii) $\frac{1}{n}\operatorname{tr}(A_n\hat{\Sigma}_n) = \frac{1}{n}\operatorname{tr}[A_n\operatorname{diag}(v_{n1}^2,\ldots,v_{nn}^2)] + o_p(1)$, and (iii) $\frac{1}{n}\operatorname{tr}[A_n\operatorname{diag}(v_{n1}^2,\ldots,v_{nn}^2)] = \frac{1}{n}\operatorname{tr}(A_n\Sigma_n) + o_p(1)$, where $A_n \equiv A_n(\phi_0) = (a_{n,ij})$.

Note that $A_n(\phi)$, $\frac{\partial A_n(\phi)}{\partial \lambda}$ and $\frac{\partial A_n(\phi)}{\partial \rho}$ are bounded in both row and column sum norms in a small neighborhood of ϕ_0, by Lemma A.3 in Lee (2004b). Then by the mean value theorem, $\frac{1}{n}\operatorname{tr}[A_n(\hat{\phi}_n)\hat{\Sigma}_n] - \frac{1}{n}\operatorname{tr}[A_n(\hat{\phi}_n)\hat{\Sigma}_n] = \frac{1}{n}\operatorname{tr}[\frac{\partial A_n(\check{\phi}_n)}{\partial \lambda}\hat{\Sigma}_n](\hat{\lambda}_n - \lambda_0) + \frac{1}{n}\operatorname{tr}[\frac{\partial A_n(\check{\phi}_n)}{\partial \rho}\hat{\Sigma}_n](\hat{\rho}_n - \rho_0)$, where $(\check{\lambda}_n, \check{\rho}_n)'$ lies between $\hat{\phi}_n$ and ϕ_0. For an $n\times n$ matrix B_n that is bounded in both row and column sum norms, $\frac{1}{n}|\operatorname{tr}(B_n\hat{\Sigma}_n)| \le \frac{c}{n}\sum_{i=1}^n \hat{v}_{ni}^2$ for some constant c. Then by (36) and Lemma 1, $\frac{1}{n}|\operatorname{tr}(B_n\hat{\Sigma}_n)| = O_p(1)$. Thus, (i) holds. For (ii), $\frac{1}{n}\operatorname{tr}(A_n\hat{\Sigma}_n) - \frac{1}{n}\operatorname{tr}[A_n\operatorname{diag}(v_{n1}^2,\ldots,v_{nn}^2)] = \frac{1}{n}\sum_{i=1}^n a_{n,ii}(\hat{v}_{ni}^2 - v_{ni}^2)$. With (36), this difference can be expanded as a polynomial of the elements of $\hat{\delta}_n - \delta_0$, whose coefficients have the form of the term in Lemma 1. Since the polynomial has no constant, (ii) holds. For (iii), $\frac{1}{n}\operatorname{tr}[A_n\operatorname{diag}(v_{n1}^2,\ldots,v_{nn}^2)] - \frac{1}{n}\operatorname{tr}(A_n\Sigma_n) = \frac{1}{n}\sum_{i=1}^n a_{n,ii}(v_{ni}^2 - \sigma_{ni}^2)$. Its variance $\frac{1}{n^2}\sum_{i=1}^n a_{n,ii}^2\,\mathrm{E}[(v_{ni}^2 - \sigma_{ni}^2)^2] = O(\frac{1}{n})$, since the disturbances are independent. Thus, (iii) holds. The proposition follows. □

Proof of Proposition 4. This proof is similar to that of Proposition 2, so it is omitted. □

Proof of Proposition 5. We only show that (21) holds, as the rest of the proof is similar to those of Propositions 1 and 2. Consider two matrices $A_n = \hat{\Omega}_n^{1/2}\binom{I_{k_1}}{-\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21}}$ and $B_n = \hat{\Omega}_n^{1/2}\binom{0}{I_{k+2-k_1}}$. It is straightforward to verify that they are orthogonal, i.e., $A_n' B_n = 0$. Since $[A_n, B_n]$ is a full rank square matrix, we have $I_{k+2} = A_n(A_n'A_n)^{-1}A_n' + B_n(B_n'B_n)^{-1}B_n'$, where the projection matrices $A_n(A_n'A_n)^{-1}A_n' = \hat{\Omega}_n^{1/2}\binom{I_{k_1}}{-\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21}}(\hat{\Omega}_{n,11} - \hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21})^{-1}[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}]\hat{\Omega}_n^{1/2}$, and $B_n(B_n'B_n)^{-1}B_n' = \hat{\Omega}_n^{1/2}\begin{pmatrix} 0_{k_1\times k_1} & 0_{k_1\times(k+2-k_1)} \\ 0_{(k+2-k_1)\times k_1} & \hat{\Omega}_{n,22}^{-1}\end{pmatrix}\hat{\Omega}_n^{1/2}$. Thus, $\binom{I_{k_1}}{-\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21}}(\hat{\Omega}_{n,11} - \hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}\hat{\Omega}_{n,21})^{-1}[I_{k_1}, -\hat{\Omega}_{n,12}\hat{\Omega}_{n,22}^{-1}] = \hat{\Omega}_n^{-1} - \begin{pmatrix} 0_{k_1\times k_1} & 0_{k_1\times(k+2-k_1)} \\ 0_{(k+2-k_1)\times k_1} & \hat{\Omega}_{n,22}^{-1}\end{pmatrix}$. Hence, (21) holds. □

Proof of Proposition 6. Similar to those of Propositions 3 and 4. □

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Table 1: Empirical sizes of tests for $\rho_0 = 0$ in the homoskedastic case

n, W_n, R^2, λ_0	S_{2SLS}	S_{GMM1}	S_{GMM2}	S_{QML}	G_{GMM1}	G_{GMM2}	D_{GMM1}	D_{GMM2}	W_{GMM1}	W_{GMM2}	W_{QML}
Normal disturbances											
144, queen, 0.4, 0.4	0.073	0.040	0.040	0.049	0.040	0.056	0.047	0.040	0.107	0.089	0.093
144, queen, 0.4, 0.8	0.123	0.034	0.042	0.044	0.041	0.053	0.048	0.051	0.070	0.059	0.072
144, queen, 0.8, 0.4	0.044	0.046	0.045	0.059	0.044	0.041	0.047	0.050	0.082	0.079	0.079
144, queen, 0.8, 0.8	0.053	0.049	0.044	0.054	0.050	0.038	0.048	0.051	0.073	0.057	0.074
144, rook, 0.4, 0.4	0.066	0.046	0.051	0.054	0.057	0.057	0.046	0.046	0.130	0.085	0.090
144, rook, 0.4, 0.8	0.106	0.039	0.054	0.057	0.050	0.060	0.059	0.066	0.088	0.067	0.068
144, rook, 0.8, 0.4	0.046	0.041	0.041	0.047	0.057	0.051	0.057	0.049	0.105	0.076	0.079
144, rook, 0.8, 0.8	0.058	0.059	0.048	0.062	0.050	0.046	0.048	0.057	0.088	0.065	0.073
400, queen, 0.4, 0.4	0.063	0.050	0.052	0.054	0.048	0.046	0.051	0.045	0.078	0.064	0.072
400, queen, 0.4, 0.8	0.069	0.039	0.044	0.043	0.045	0.046	0.045	0.050	0.045	0.047	0.049
400, queen, 0.8, 0.4	0.044	0.038	0.040	0.041	0.041	0.042	0.040	0.041	0.054	0.045	0.050
400, queen, 0.8, 0.8	0.059	0.059	0.055	0.065	0.071	0.056	0.063	0.065	0.089	0.075	0.068
400, rook, 0.4, 0.4	0.052	0.040	0.039	0.041	0.044	0.044	0.038	0.036	0.079	0.053	0.056
400, rook, 0.4, 0.8	0.050	0.041	0.043	0.043	0.040	0.040	0.043	0.043	0.061	0.042	0.042
400, rook, 0.8, 0.4	0.047	0.051	0.050	0.053	0.054	0.056	0.051	0.049	0.065	0.058	0.063
400, rook, 0.8, 0.8	0.052	0.050	0.049	0.054	0.056	0.055	0.057	0.053	0.080	0.070	0.062
Chi-squared disturbances											
144, queen, 0.4, 0.4	0.082	0.034	0.034	0.042	0.041	0.040	0.042	0.044	0.117	0.094	0.102
144, queen, 0.4, 0.8	0.105	0.029	0.038	0.050	0.043	0.040	0.043	0.047	0.086	0.067	0.076
144, queen, 0.8, 0.4	0.060	0.050	0.045	0.067	0.046	0.051	0.049	0.055	0.107	0.098	0.107
144, queen, 0.8, 0.8	0.057	0.042	0.039	0.060	0.043	0.040	0.040	0.044	0.082	0.072	0.092
144, rook, 0.4, 0.4	0.058	0.036	0.038	0.047	0.061	0.046	0.050	0.052	0.130	0.095	0.102
144, rook, 0.4, 0.8	0.084	0.029	0.040	0.043	0.044	0.048	0.042	0.040	0.098	0.070	0.073
144, rook, 0.8, 0.4	0.051	0.056	0.054	0.064	0.055	0.056	0.054	0.049	0.103	0.097	0.103
144, rook, 0.8, 0.8	0.057	0.051	0.050	0.062	0.063	0.051	0.050	0.048	0.088	0.075	0.096
400, queen, 0.4, 0.4	0.061	0.047	0.049	0.053	0.053	0.045	0.049	0.054	0.076	0.066	0.072
400, queen, 0.4, 0.8	0.081	0.049	0.051	0.053	0.051	0.053	0.055	0.053	0.072	0.065	0.064
400, queen, 0.8, 0.4	0.042	0.040	0.041	0.044	0.048	0.045	0.045	0.040	0.063	0.059	0.054
400, queen, 0.8, 0.8	0.049	0.049	0.047	0.050	0.050	0.046	0.047	0.050	0.064	0.065	0.060
400, rook, 0.4, 0.4	0.068	0.058	0.057	0.059	0.059	0.061	0.060	0.052	0.097	0.076	0.082
400, rook, 0.4, 0.8	0.057	0.044	0.044	0.045	0.041	0.046	0.037	0.047	0.077	0.051	0.054
400, rook, 0.8, 0.4	0.050	0.048	0.053	0.057	0.052	0.056	0.050	0.051	0.081	0.069	0.072
400, rook, 0.8, 0.8	0.045	0.043	0.045	0.045	0.046	0.046	0.041	0.045	0.066	0.052	0.059

(1) S_{2SLS}, S_{GMM1}, S_{GMM2} and S_{QML} denote score-based OPG tests with, respectively, the 2SLS, GMM1, GMM2 and QML estimates of the SAR model;

(2) G_{GMM1} and G_{GMM2} denote gradient-based OPG tests with, respectively, the GMM1 and GMM2 estimates of the SAR model;

(3) D_{GMM1} and D_{GMM2} denote distance difference tests based on different moment vectors;

(4) W_{GMM1} and W_{GMM2} denote Wald tests based on the GMM estimation of the SARAR model with different moment vectors;

(5) W_{QML} denotes the Wald test based on the QML estimation of the SARAR model.

Table 2: Empirical sizes of tests for $\rho_0 = 0$ in the heteroskedastic case

n, W_n, R^2, λ_0	S_{2SLS}	S_{RGMM1}	S_{RGMM2}	S_{MQML}	G_{RGMM1}	G_{RGMM2}	D_{RGMM1}	D_{RGMM2}	W_{RGMM1}	W_{RGMM2}	W_{MQML}
Normal disturbances											
144, queen, 0.4, 0.4	0.056	0.044	0.045	0.061	0.044	0.040	0.045	0.044	0.128	0.111	0.098
144, queen, 0.4, 0.8	0.078	0.035	0.046	0.055	0.051	0.050	0.057	0.051	0.154	0.138	0.130
144, queen, 0.8, 0.4	0.056	0.030	0.026	0.044	0.036	0.025	0.052	0.039	0.113	0.084	0.071
144, queen, 0.8, 0.8	0.075	0.031	0.025	0.041	0.033	0.031	0.049	0.034	0.086	0.068	0.068
144, rook, 0.4, 0.4	0.053	0.048	0.044	0.058	0.052	0.044	0.045	0.048	0.141	0.114	0.089
144, rook, 0.4, 0.8	0.102	0.028	0.036	0.049	0.043	0.037	0.047	0.045	0.147	0.142	0.092
144, rook, 0.8, 0.4	0.042	0.035	0.035	0.040	0.034	0.034	0.043	0.049	0.115	0.081	0.070
144, rook, 0.8, 0.8	0.057	0.037	0.037	0.047	0.038	0.035	0.048	0.043	0.103	0.080	0.066
400, queen, 0.4, 0.4	0.042	0.034	0.026	0.033	0.039	0.033	0.044	0.037	0.100	0.073	0.059
400, queen, 0.4, 0.8	0.075	0.048	0.045	0.054	0.037	0.054	0.048	0.049	0.151	0.147	0.063
400, queen, 0.8, 0.4	0.044	0.025	0.026	0.032	0.036	0.031	0.039	0.038	0.074	0.066	0.046
400, queen, 0.8, 0.8	0.041	0.033	0.034	0.040	0.035	0.033	0.035	0.038	0.066	0.058	0.050
400, rook, 0.4, 0.4	0.044	0.044	0.045	0.051	0.049	0.051	0.052	0.048	0.097	0.077	0.071
400, rook, 0.4, 0.8	0.060	0.051	0.052	0.053	0.049	0.055	0.047	0.048	0.116	0.113	0.061
400, rook, 0.8, 0.4	0.041	0.036	0.035	0.043	0.036	0.034	0.038	0.044	0.085	0.068	0.054
400, rook, 0.8, 0.8	0.041	0.049	0.049	0.057	0.042	0.048	0.046	0.057	0.085	0.077	0.063
Chi-squared disturbances											
144, queen, 0.4, 0.4	0.054	0.033	0.032	0.048	0.057	0.039	0.040	0.031	0.150	0.112	0.122
144, queen, 0.4, 0.8	0.079	0.032	0.043	0.053	0.036	0.052	0.032	0.033	0.147	0.137	0.112
144, queen, 0.8, 0.4	0.077	0.039	0.037	0.059	0.043	0.037	0.041	0.041	0.120	0.106	0.093
144, queen, 0.8, 0.8	0.077	0.037	0.040	0.051	0.036	0.037	0.039	0.036	0.090	0.069	0.067
144, rook, 0.4, 0.4	0.044	0.028	0.032	0.042	0.045	0.042	0.043	0.039	0.138	0.103	0.111
144, rook, 0.4, 0.8	0.065	0.032	0.040	0.047	0.047	0.041	0.047	0.037	0.146	0.119	0.094
144, rook, 0.8, 0.4	0.061	0.030	0.032	0.040	0.052	0.040	0.033	0.037	0.103	0.089	0.093
144, rook, 0.8, 0.8	0.050	0.020	0.030	0.037	0.045	0.032	0.040	0.028	0.097	0.091	0.074
400, queen, 0.4, 0.4	0.052	0.042	0.048	0.049	0.056	0.045	0.050	0.048	0.099	0.085	0.081
400, queen, 0.4, 0.8	0.048	0.037	0.040	0.044	0.046	0.045	0.049	0.039	0.136	0.126	0.065
400, queen, 0.8, 0.4	0.069	0.045	0.044	0.062	0.048	0.047	0.040	0.047	0.088	0.089	0.083
400, queen, 0.8, 0.8	0.051	0.035	0.034	0.045	0.047	0.034	0.042	0.037	0.084	0.064	0.058
400, rook, 0.4, 0.4	0.058	0.044	0.047	0.052	0.049	0.052	0.043	0.051	0.081	0.070	0.084
400, rook, 0.4, 0.8	0.062	0.043	0.045	0.048	0.057	0.044	0.056	0.049	0.131	0.134	0.055
400, rook, 0.8, 0.4	0.053	0.039	0.043	0.047	0.046	0.043	0.043	0.049	0.080	0.074	0.071
400, rook, 0.8, 0.8	0.039	0.034	0.038	0.041	0.042	0.041	0.034	0.038	0.085	0.076	0.062

(1) S_{2SLS}, S_{RGMM1}, S_{RGMM2} and S_{MQML} denote score-based OPG tests with, respectively, the 2SLS, RGMM1, RGMM2 and MQML estimates of the SAR model;

(2) G_{RGMM1} and G_{RGMM2} denote gradient-based OPG tests with, respectively, the RGMM1 and RGMM2 estimates of the SAR model;

(3) D_{RGMM1} and D_{RGMM2} denote distance difference tests based on different moment vectors;

(4) W_{RGMM1} and W_{RGMM2} denote Wald tests based on the robust GMM estimation of the SARAR model with different moment vectors;

(5) W_{MQML} denotes the Wald test based on the MQML estimation of the SARAR model.

Table 3: Empirical sizes of tests for $\lambda_0 = 0$ in the homoskedastic case

n, W_n, R^2, ρ_0	S_{LS}	S_{GMM1}	S_{GMM2}	S_{QML}	G_{GMM1}	G_{GMM2}	D_{GMM1}	D_{GMM2}	W_{GMM1}	W_{GMM2}	W_{QML}
Normal disturbances											
144, queen, 0.4, 0.4	0.066	0.056	0.059	0.058	0.053	0.052	0.045	0.037	0.145	0.125	0.110
144, queen, 0.4, 0.8	0.045	0.033	0.046	0.042	0.056	0.053	0.056	0.052	0.118	0.109	0.116
144, queen, 0.8, 0.4	0.054	0.051	0.056	0.051	0.055	0.063	0.057	0.050	0.110	0.095	0.086
144, queen, 0.8, 0.8	0.054	0.047	0.054	0.053	0.047	0.047	0.042	0.048	0.107	0.077	0.085
144, rook, 0.4, 0.4	0.049	0.046	0.048	0.048	0.053	0.052	0.045	0.041	0.149	0.123	0.107
144, rook, 0.4, 0.8	0.035	0.040	0.048	0.045	0.046	0.047	0.058	0.052	0.082	0.069	0.064
144, rook, 0.8, 0.4	0.069	0.060	0.063	0.063	0.060	0.068	0.052	0.053	0.101	0.079	0.077
144, rook, 0.8, 0.8	0.036	0.044	0.041	0.041	0.041	0.040	0.036	0.036	0.078	0.053	0.050
400, queen, 0.4, 0.4	0.041	0.040	0.040	0.040	0.041	0.045	0.043	0.042	0.086	0.086	0.073
400, queen, 0.4, 0.8	0.067	0.069	0.075	0.074	0.054	0.055	0.061	0.066	0.080	0.074	0.081
400, queen, 0.8, 0.4	0.047	0.048	0.046	0.046	0.042	0.046	0.040	0.044	0.064	0.058	0.056
400, queen, 0.8, 0.8	0.052	0.055	0.056	0.056	0.057	0.056	0.056	0.049	0.076	0.077	0.071
400, rook, 0.4, 0.4	0.052	0.054	0.055	0.054	0.056	0.053	0.047	0.043	0.091	0.072	0.075
400, rook, 0.4, 0.8	0.044	0.040	0.043	0.042	0.045	0.047	0.042	0.043	0.055	0.038	0.043
400, rook, 0.8, 0.4	0.048	0.047	0.047	0.047	0.053	0.048	0.055	0.055	0.065	0.060	0.056
400, rook, 0.8, 0.8	0.052	0.055	0.056	0.056	0.056	0.051	0.052	0.049	0.068	0.060	0.058
Chi-squared disturbances											
144, queen, 0.4, 0.4	0.058	0.038	0.058	0.058	0.037	0.058	0.027	0.029	0.163	0.125	0.110
144, queen, 0.4, 0.8	0.051	0.029	0.058	0.053	0.051	0.050	0.052	0.049	0.133	0.121	0.126
144, queen, 0.8, 0.4	0.048	0.039	0.046	0.046	0.050	0.052	0.048	0.042	0.085	0.075	0.073
144, queen, 0.8, 0.8	0.053	0.053	0.050	0.050	0.052	0.052	0.039	0.045	0.096	0.081	0.081
144, rook, 0.4, 0.4	0.052	0.039	0.051	0.050	0.050	0.054	0.042	0.041	0.134	0.101	0.081
144, rook, 0.4, 0.8	0.048	0.035	0.064	0.063	0.054	0.057	0.060	0.064	0.119	0.083	0.088
144, rook, 0.8, 0.4	0.055	0.048	0.056	0.056	0.059	0.051	0.055	0.048	0.091	0.077	0.064
144, rook, 0.8, 0.8	0.043	0.042	0.052	0.050	0.045	0.049	0.043	0.044	0.071	0.056	0.063
400, queen, 0.4, 0.4	0.049	0.047	0.045	0.045	0.058	0.050	0.051	0.045	0.104	0.084	0.080
400, queen, 0.4, 0.8	0.056	0.040	0.056	0.056	0.053	0.055	0.059	0.051	0.086	0.069	0.067
400, queen, 0.8, 0.4	0.049	0.049	0.051	0.049	0.051	0.048	0.051	0.050	0.064	0.058	0.063
400, queen, 0.8, 0.8	0.051	0.047	0.052	0.052	0.048	0.050	0.046	0.044	0.075	0.067	0.058
400, rook, 0.4, 0.4	0.037	0.031	0.033	0.033	0.044	0.037	0.036	0.035	0.072	0.058	0.049
400, rook, 0.4, 0.8	0.051	0.051	0.063	0.063	0.048	0.045	0.054	0.066	0.073	0.075	0.065
400, rook, 0.8, 0.4	0.049	0.049	0.049	0.049	0.046	0.047	0.048	0.043	0.058	0.047	0.055
400, rook, 0.8, 0.8	0.044	0.046	0.048	0.049	0.055	0.052	0.050	0.041	0.060	0.051	0.050

(1) S_{2SLS}, S_{GMM1}, S_{GMM2} and S_{QML} denote score-based OPG tests with, respectively, the 2SLS, GMM1, GMM2 and QML estimates of the SAR model;

(2) G_{GMM1} and G_{GMM2} denote gradient-based OPG tests with, respectively, the GMM1 and GMM2 estimates of the SAR model;

(3) D_{GMM1} and D_{GMM2} denote distance difference tests based on different moment vectors;

(4) W_{GMM1} and W_{GMM2} denote Wald tests based on the GMM estimation of the SARAR model with different moment vectors;

(5) W_{QML} denotes the Wald test based on the QML estimation of the SARAR model.

Table 4: Empirical sizes of tests for $\lambda_0 = 0$ in the heteroskedastic case

n, W_n, R^2, ρ_0	S_{LS}	S_{RGMM1}	S_{RGMM2}	S_{MQML}	G_{RGMM1}	G_{RGMM2}	D_{RGMM1}	D_{RGMM2}	W_{RGMM1}	W_{RGMM2}	W_{MQML}
Normal disturbances											
144, queen, 0.4, 0.4	0.063	0.036	0.057	0.057	0.048	0.059	0.049	0.046	0.165	0.150	0.131
144, queen, 0.4, 0.8	0.051	0.034	0.063	0.057	0.040	0.058	0.057	0.059	0.220	0.185	0.158
144, queen, 0.8, 0.4	0.051	0.034	0.051	0.049	0.050	0.056	0.051	0.043	0.110	0.081	0.078
144, queen, 0.8, 0.8	0.059	0.052	0.057	0.055	0.052	0.049	0.059	0.053	0.130	0.096	0.085
144, rook, 0.4, 0.4	0.046	0.033	0.042	0.042	0.040	0.050	0.047	0.038	0.137	0.110	0.093
144, rook, 0.4, 0.8	0.041	0.042	0.052	0.052	0.043	0.054	0.049	0.048	0.224	0.205	0.130
144, rook, 0.8, 0.4	0.050	0.039	0.049	0.048	0.049	0.049	0.049	0.043	0.097	0.074	0.067
144, rook, 0.8, 0.8	0.058	0.049	0.055	0.056	0.048	0.059	0.054	0.052	0.102	0.093	0.074
400, queen, 0.4, 0.4	0.049	0.045	0.046	0.045	0.051	0.046	0.045	0.041	0.112	0.096	0.091
400, queen, 0.4, 0.8	0.047	0.044	0.055	0.051	0.053	0.066	0.062	0.059	0.136	0.132	0.090
400, queen, 0.8, 0.4	0.047	0.043	0.044	0.043	0.044	0.045	0.040	0.037	0.056	0.052	0.056
400, queen, 0.8, 0.8	0.043	0.034	0.042	0.039	0.050	0.039	0.056	0.040	0.076	0.067	0.056
400, rook, 0.4, 0.4	0.051	0.040	0.053	0.051	0.044	0.049	0.045	0.039	0.090	0.076	0.080
400, rook, 0.4, 0.8	0.051	0.049	0.059	0.058	0.056	0.046	0.058	0.058	0.132	0.128	0.080
400, rook, 0.8, 0.4	0.053	0.045	0.051	0.051	0.048	0.051	0.057	0.047	0.064	0.060	0.058
400, rook, 0.8, 0.8	0.046	0.036	0.040	0.038	0.042	0.049	0.046	0.056	0.070	0.068	0.051
Chi-squared disturbances											
144, queen, 0.4, 0.4	0.053	0.034	0.054	0.055	0.042	0.061	0.046	0.034	0.175	0.136	0.145
144, queen, 0.4, 0.8	0.035	0.025	0.040	0.038	0.055	0.048	0.043	0.034	0.158	0.141	0.135
144, queen, 0.8, 0.4	0.051	0.035	0.047	0.046	0.050	0.048	0.046	0.036	0.092	0.085	0.085
144, queen, 0.8, 0.8	0.054	0.044	0.053	0.052	0.064	0.059	0.041	0.046	0.096	0.098	0.090
144, rook, 0.4, 0.4	0.029	0.027	0.026	0.026	0.052	0.033	0.043	0.037	0.160	0.114	0.108
144, rook, 0.4, 0.8	0.047	0.033	0.042	0.039	0.064	0.055	0.038	0.041	0.194	0.180	0.111
144, rook, 0.8, 0.4	0.052	0.041	0.054	0.053	0.059	0.055	0.052	0.052	0.099	0.078	0.075
144, rook, 0.8, 0.8	0.057	0.049	0.056	0.055	0.074	0.064	0.062	0.051	0.108	0.094	0.078
400, queen, 0.4, 0.4	0.041	0.042	0.040	0.041	0.050	0.041	0.037	0.040	0.088	0.078	0.102
400, queen, 0.4, 0.8	0.048	0.044	0.052	0.048	0.053	0.057	0.048	0.048	0.108	0.089	0.074
400, queen, 0.8, 0.4	0.058	0.057	0.056	0.056	0.067	0.060	0.059	0.059	0.077	0.082	0.076
400, queen, 0.8, 0.8	0.051	0.048	0.050	0.050	0.060	0.056	0.055	0.045	0.079	0.072	0.066
400, rook, 0.4, 0.4	0.040	0.043	0.042	0.041	0.055	0.048	0.036	0.044	0.092	0.080	0.067
400, rook, 0.4, 0.8	0.044	0.042	0.047	0.047	0.058	0.053	0.046	0.043	0.122	0.127	0.067
400, rook, 0.8, 0.4	0.060	0.054	0.058	0.057	0.068	0.065	0.060	0.063	0.078	0.073	0.067
400, rook, 0.8, 0.8	0.048	0.045	0.045	0.043	0.058	0.049	0.064	0.053	0.081	0.075	0.053

(1) S_{2SLS}, S_{RGMM1}, S_{RGMM2} and S_{MQML} denote score-based OPG tests with, respectively, the 2SLS, RGMM1, RGMM2 and MQML estimates of the SAR model;

(2) G_{RGMM1} and G_{RGMM2} denote gradient-based OPG tests with, respectively, the RGMM1 and RGMM2 estimates of the SAR model;

(3) D_{RGMM1} and D_{RGMM2} denote distance difference tests based on different moment vectors;

(4) W_{RGMM1} and W_{RGMM2} denote Wald tests based on the robust GMM estimation of the SARAR model with different moment vectors;

(5) W_{MQML} denotes the Wald test based on the MQML estimation of the SARAR model.

Table 5: Powers of tests for $\rho_0 = 0$ with normal homoskedastic disturbances

n, W_n, R^2, λ_0, ρ_0	S_{2SLS}	S_{GMM1}	S_{GMM2}	S_{QML}	G_{GMM1}	G_{GMM2}	D_{GMM1}	D_{GMM2}	W_{GMM1}	W_{GMM2}	W_{QML}
144, queen, 0.4, 0.4, 0.2	0.182	0.083	0.089	0.104	0.085	0.097	0.081	0.081	0.153	0.116	0.159
144, queen, 0.4, 0.4, 0.4	0.384	0.215	0.240	0.259	0.200	0.228	0.217	0.232	0.285	0.236	0.291
144, queen, 0.4, 0.4, 0.6	0.542	0.357	0.416	0.457	0.327	0.385	0.472	0.480	0.495	0.434	0.506
144, queen, 0.4, 0.4, 0.8	0.590	0.547	0.643	0.698	0.463	0.601	0.777	0.764	0.705	0.630	0.736
144, queen, 0.4, 0.8, 0.2	0.203	0.105	0.133	0.137	0.120	0.141	0.147	0.156	0.109	0.091	0.112
144, queen, 0.4, 0.8, 0.4	0.431	0.346	0.413	0.434	0.302	0.410	0.427	0.464	0.283	0.243	0.322
144, queen, 0.4, 0.8, 0.6	0.624	0.740	0.801	0.815	0.584	0.780	0.804	0.828	0.572	0.474	0.712
144, queen, 0.4, 0.8, 0.8	0.778	0.913	0.961	0.975	0.807	0.944	0.966	0.948	0.818	0.687	0.962
144, queen, 0.8, 0.4, 0.2	0.188	0.149	0.146	0.159	0.139	0.150	0.140	0.151	0.171	0.162	0.187
144, queen, 0.8, 0.4, 0.4	0.558	0.502	0.496	0.516	0.429	0.477	0.452	0.481	0.505	0.506	0.558
144, queen, 0.8, 0.4, 0.6	0.895	0.855	0.857	0.871	0.779	0.820	0.842	0.855	0.866	0.864	0.904
144, queen, 0.8, 0.4, 0.8	0.990	0.986	0.981	0.988	0.920	0.929	0.982	0.980	0.989	0.985	0.994
144, queen, 0.8, 0.8, 0.2	0.191	0.153	0.155	0.160	0.129	0.150	0.142	0.171	0.157	0.147	0.188
144, queen, 0.8, 0.8, 0.4	0.611	0.541	0.556	0.572	0.475	0.525	0.499	0.557	0.516	0.530	0.595
144, queen, 0.8, 0.8, 0.6	0.924	0.895	0.900	0.905	0.819	0.866	0.867	0.889	0.879	0.884	0.921
144, queen, 0.8, 0.8, 0.8	0.985	0.987	0.982	0.992	0.944	0.957	0.982	0.986	0.981	0.985	0.994
144, rook, 0.4, 0.4, 0.2	0.181	0.152	0.163	0.176	0.185	0.169	0.152	0.148	0.179	0.144	0.184
144, rook, 0.4, 0.4, 0.4	0.451	0.369	0.405	0.434	0.386	0.394	0.385	0.393	0.373	0.305	0.399
144, rook, 0.4, 0.4, 0.6	0.691	0.564	0.616	0.659	0.585	0.583	0.738	0.724	0.695	0.619	0.728
144, rook, 0.4, 0.4, 0.8	0.647	0.684	0.785	0.849	0.675	0.754	0.922	0.914	0.902	0.857	0.925
144, rook, 0.4, 0.8, 0.2	0.314	0.228	0.269	0.279	0.242	0.265	0.272	0.284	0.174	0.146	0.219
144, rook, 0.4, 0.8, 0.4	0.632	0.607	0.680	0.701	0.550	0.672	0.670	0.710	0.484	0.445	0.603
144, rook, 0.4, 0.8, 0.6	0.816	0.914	0.946	0.963	0.838	0.942	0.949	0.962	0.837	0.790	0.948
144, rook, 0.4, 0.8, 0.8	0.864	0.955	0.976	1.000	0.912	0.962	0.991	0.977	0.982	0.924	0.999
144, rook, 0.8, 0.4, 0.2	0.251	0.238	0.240	0.251	0.260	0.245	0.225	0.235	0.249	0.235	0.295
144, rook, 0.8, 0.4, 0.4	0.788	0.756	0.755	0.771	0.732	0.730	0.736	0.753	0.780	0.773	0.818
144, rook, 0.8, 0.4, 0.6	0.987	0.967	0.964	0.971	0.924	0.911	0.964	0.964	0.976	0.966	0.992
144, rook, 0.8, 0.4, 0.8	0.999	0.990	0.972	0.994	0.941	0.845	0.997	0.993	0.999	0.993	1.000
144, rook, 0.8, 0.8, 0.2	0.354	0.321	0.327	0.336	0.307	0.314	0.267	0.325	0.285	0.300	0.360
144, rook, 0.8, 0.8, 0.4	0.818	0.783	0.789	0.803	0.759	0.765	0.754	0.801	0.751	0.780	0.825
144, rook, 0.8, 0.8, 0.6	0.985	0.981	0.980	0.984	0.951	0.968	0.963	0.980	0.965	0.976	0.986
144, rook, 0.8, 0.8, 0.8	0.993	0.999	0.991	1.000	0.968	0.976	0.996	0.991	1.000	0.997	1.000
400, queen, 0.4, 0.4, 0.2	0.293	0.201	0.204	0.215	0.193	0.194	0.205	0.206	0.211	0.184	0.219
400, queen, 0.4, 0.4, 0.4	0.671	0.559	0.571	0.577	0.532	0.553	0.581	0.594	0.539	0.502	0.558
400, queen, 0.4, 0.4, 0.6	0.914	0.855	0.863	0.874	0.826	0.847	0.892	0.894	0.870	0.844	0.885
400, queen, 0.4, 0.4, 0.8	0.923	0.942	0.959	0.970	0.909	0.949	0.981	0.973	0.979	0.974	0.987
400, queen, 0.4, 0.8, 0.2	0.383	0.289	0.316	0.325	0.261	0.312	0.299	0.340	0.225	0.233	0.270
400, queen, 0.4, 0.8, 0.4	0.818	0.780	0.810	0.811	0.713	0.804	0.774	0.826	0.690	0.739	0.791
400, queen, 0.4, 0.8, 0.6	0.950	0.982	0.993	0.993	0.959	0.991	0.985	0.994	0.975	0.983	0.993
400, queen, 0.4, 0.8, 0.8	0.919	0.999	1.000	1.000	0.993	1.000	1.000	0.999	1.000	0.996	1.000
400, queen, 0.8, 0.4, 0.2	0.403	0.363	0.366	0.369	0.348	0.358	0.355	0.347	0.368	0.354	0.398
400, queen, 0.8, 0.4, 0.4	0.931	0.916	0.914	0.921	0.909	0.908	0.910	0.912	0.918	0.912	0.929
400, queen, 0.8, 0.4, 0.6	0.998	0.998	0.998	0.998	0.998	0.998	0.998	0.997	0.998	0.997	0.999
400, queen, 0.8, 0.4, 0.8	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.2	0.481	0.442	0.447	0.449	0.393	0.409	0.413	0.447	0.392	0.431	0.453
400, queen, 0.8, 0.8, 0.4	0.958	0.948	0.947	0.948	0.929	0.941	0.933	0.952	0.923	0.943	0.952
400, queen, 0.8, 0.8, 0.6	0.999	0.999	0.999	0.999	0.996	0.998	0.997	0.999	0.997	0.999	0.999
400, queen, 0.8, 0.8, 0.8	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.4, 0.4, 0.2	0.354	0.334	0.338	0.342	0.336	0.337	0.319	0.348	0.288	0.255	0.313
400, rook, 0.4, 0.4, 0.4	0.857	0.825	0.832	0.841	0.813	0.814	0.853	0.861	0.795	0.760	0.838
400, rook, 0.4, 0.4, 0.6	0.970	0.973	0.977	0.982	0.967	0.971	0.986	0.986	0.983	0.974	0.993
400, rook, 0.4, 0.4, 0.8	0.618	0.979	0.991	0.995	0.984	0.983	0.995	0.994	0.995	0.995	0.999
400, rook, 0.4, 0.8, 0.2	0.508	0.486	0.506	0.506	0.466	0.503	0.478	0.538	0.365	0.431	0.467
400, rook, 0.4, 0.8, 0.4	0.928	0.963	0.974	0.974	0.939	0.975	0.958	0.976	0.920	0.956	0.969
400, rook, 0.4, 0.8, 0.6	0.908	1.000	0.999	1.000	0.998	0.999	1.000	1.000	0.999	1.000	1.000
400, rook, 0.4, 0.8, 0.8	0.858	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.2	0.642	0.637	0.635	0.641	0.638	0.634	0.609	0.622	0.603	0.610	0.656
400, rook, 0.8, 0.4, 0.4	0.993	0.988	0.988	0.989	0.987	0.986	0.989	0.990	0.991	0.989	0.992
400, rook, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	1.000	1.000	1.000	1.000	0.998	0.997	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.2	0.703	0.686	0.683	0.691	0.673	0.678	0.646	0.693	0.608	0.666	0.692
400, rook, 0.8, 0.8, 0.4	0.997	0.996	0.996	0.996	0.997	0.995	0.996	0.997	0.993	0.997	0.998
400, rook, 0.8, 0.8, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.993	1.000	0.999	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000

(1) S_{2SLS}, S_{GMM1}, S_{GMM2} and S_{QML} denote score-based OPG tests with, respectively, the 2SLS, GMM1, GMM2 and QML estimates of the SAR model;
(2) G_{GMM1} and G_{GMM2} denote gradient-based OPG tests with, respectively, the GMM1 and GMM2 estimates of the SAR model;
(3) D_{GMM1} and D_{GMM2} denote distance difference tests based on different moment vectors;
(4) W_{GMM1} and W_{GMM2} denote Wald tests based on the GMM estimation of the SARAR model with different moment vectors;
(5) W_{QML} denotes the Wald test based on the QML estimation of the SARAR model.

Table 6: Powers of tests for $\rho_0 = 0$ with chi-squared homoskedastic disturbances

n, W_n, R^2, λ_0, ρ_0	S_{2SLS}	S_{GMM1}	S_{GMM2}	S_{QML}	G_{GMM1}	G_{GMM2}	D_{GMM1}	D_{GMM2}	W_{GMM1}	W_{GMM2}	W_{QML}
144, queen, 0.4, 0.4, 0.2	0.185	0.079	0.093	0.110	0.114	0.095	0.112	0.104	0.167	0.135	0.166
144, queen, 0.4, 0.4, 0.4	0.380	0.212	0.245	0.268	0.226	0.237	0.282	0.290	0.357	0.304	0.312
144, queen, 0.4, 0.4, 0.6	0.587	0.365	0.458	0.518	0.378	0.441	0.551	0.546	0.571	0.494	0.538
144, queen, 0.4, 0.4, 0.8	0.627	0.530	0.636	0.719	0.506	0.600	0.836	0.793	0.793	0.710	0.770
144, queen, 0.4, 0.8, 0.2	0.222	0.092	0.135	0.153	0.116	0.132	0.158	0.158	0.122	0.108	0.127
144, queen, 0.4, 0.8, 0.4	0.441	0.310	0.428	0.447	0.298	0.414	0.446	0.443	0.321	0.267	0.339
144, queen, 0.4, 0.8, 0.6	0.616	0.682	0.782	0.803	0.592	0.748	0.818	0.811	0.601	0.533	0.702
144, queen, 0.4, 0.8, 0.8	0.780	0.909	0.962	0.985	0.783	0.939	0.974	0.961	0.860	0.753	0.961
144, queen, 0.8, 0.4, 0.2	0.162	0.137	0.130	0.148	0.120	0.137	0.137	0.138	0.186	0.156	0.191
144, queen, 0.8, 0.4, 0.4	0.591	0.544	0.533	0.545	0.486	0.504	0.534	0.554	0.580	0.557	0.583
144, queen, 0.8, 0.4, 0.6	0.912	0.873	0.872	0.885	0.814	0.828	0.889	0.884	0.909	0.885	0.918
144, queen, 0.8, 0.4, 0.8	0.982	0.976	0.972	0.979	0.897	0.913	0.980	0.974	0.987	0.978	0.989
144, queen, 0.8, 0.8, 0.2	0.184	0.144	0.151	0.155	0.130	0.145	0.147	0.160	0.156	0.142	0.193
144, queen, 0.8, 0.8, 0.4	0.655	0.597	0.608	0.616	0.517	0.579	0.567	0.590	0.580	0.562	0.638
144, queen, 0.8, 0.8, 0.6	0.938	0.910	0.915	0.928	0.832	0.892	0.899	0.928	0.899	0.903	0.943
144, queen, 0.8, 0.8, 0.8	0.990	0.993	0.988	0.996	0.909	0.947	0.989	0.986	0.989	0.988	0.996
144, rook, 0.4, 0.4, 0.2	0.205	0.150	0.159	0.177	0.196	0.169	0.154	0.156	0.212	0.165	0.200
144, rook, 0.4, 0.4, 0.4	0.505	0.365	0.422	0.463	0.443	0.399	0.462	0.455	0.474	0.384	0.452
144, rook, 0.4, 0.4, 0.6	0.729	0.582	0.667	0.723	0.604	0.629	0.789	0.775	0.758	0.677	0.744
144, rook, 0.4, 0.4, 0.8	0.695	0.637	0.760	0.856	0.650	0.730	0.938	0.915	0.914	0.863	0.920
144, rook, 0.4, 0.8, 0.2	0.276	0.150	0.233	0.255	0.215	0.227	0.265	0.257	0.184	0.149	0.204
144, rook, 0.4, 0.8, 0.4	0.635	0.525	0.671	0.704	0.546	0.656	0.690	0.686	0.519	0.475	0.627
144, rook, 0.4, 0.8, 0.6	0.822	0.891	0.957	0.973	0.815	0.936	0.962	0.971	0.865	0.824	0.944
144, rook, 0.4, 0.8, 0.8	0.890	0.964	0.975	0.999	0.886	0.954	0.996	0.966	0.989	0.919	0.995
144, rook, 0.8, 0.4, 0.2	0.272	0.264	0.257	0.273	0.295	0.268	0.265	0.267	0.296	0.279	0.320
144, rook, 0.8, 0.4, 0.4	0.798	0.761	0.754	0.782	0.750	0.730	0.765	0.773	0.795	0.780	0.822
144, rook, 0.8, 0.4, 0.6	0.989	0.973	0.970	0.976	0.941	0.924	0.983	0.979	0.991	0.981	0.992
144, rook, 0.8, 0.4, 0.8	0.998	0.987	0.978	0.996	0.905	0.857	0.999	0.997	1.000	0.995	0.997
144, rook, 0.8, 0.8, 0.2	0.306	0.273	0.277	0.288	0.287	0.281	0.278	0.295	0.279	0.266	0.322
144, rook, 0.8, 0.8, 0.4	0.863	0.822	0.831	0.846	0.788	0.802	0.792	0.846	0.799	0.804	0.871
144, rook, 0.8, 0.8, 0.6	0.994	0.992	0.989	0.993	0.944	0.971	0.981	0.990	0.988	0.991	0.995
144, rook, 0.8, 0.8, 0.8	0.993	0.996	0.992	1.000	0.910	0.972	0.999	0.998	0.999	0.997	1.000
400, queen, 0.4, 0.4, 0.2	0.276	0.203	0.205	0.210	0.212	0.206	0.201	0.203	0.175	0.151	0.174
400, queen, 0.4, 0.4, 0.4	0.661	0.557	0.563	0.579	0.551	0.551	0.583	0.592	0.543	0.504	0.564
400, queen, 0.4, 0.4, 0.6	0.897	0.824	0.835	0.850	0.806	0.809	0.886	0.890	0.867	0.836	0.877
400, queen, 0.4, 0.4, 0.8	0.921	0.937	0.951	0.969	0.911	0.940	0.980	0.980	0.972	0.976	0.983
400, queen, 0.4, 0.8, 0.2	0.351	0.249	0.283	0.298	0.253	0.283	0.294	0.301	0.203	0.210	0.255
400, queen, 0.4, 0.8, 0.4	0.825	0.766	0.813	0.820	0.726	0.811	0.788	0.830	0.705	0.744	0.803
400, queen, 0.4, 0.8, 0.6	0.961	0.991	0.997	0.999	0.965	0.996	0.991	0.999	0.981	0.992	0.998
400, queen, 0.4, 0.8, 0.8	0.940	0.997	1.000	1.000	0.997	0.999	1.000	0.998	1.000	0.994	1.000
400, queen, 0.8, 0.4, 0.2	0.433	0.399	0.394	0.401	0.369	0.377	0.394	0.394	0.406	0.394	0.425
400, queen, 0.8, 0.4, 0.4	0.952	0.940	0.939	0.941	0.932	0.929	0.942	0.942	0.954	0.943	0.954
400, queen, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.4, 0.8	1.000	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.2	0.454	0.416	0.416	0.418	0.375	0.392	0.385	0.420	0.354	0.392	0.432
400, queen, 0.8, 0.8, 0.4	0.970	0.960	0.960	0.962	0.942	0.956	0.952	0.958	0.936	0.953	0.961
400, queen, 0.8, 0.8, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.998	1.000
400, queen, 0.8, 0.8, 0.8	1.000	1.000	1.000	1.000	0.997	0.996	1.000	1.000	1.000	1.000	1.000
400, rook, 0.4, 0.4, 0.2	0.351	0.329	0.334	0.337	0.356	0.329	0.365	0.362	0.309	0.273	0.329
400, rook, 0.4, 0.4, 0.4	0.819	0.798	0.804	0.813	0.819	0.805	0.844	0.845	0.798	0.762	0.809
400, rook, 0.4, 0.4, 0.6	0.958	0.969	0.973	0.974	0.972	0.971	0.991	0.986	0.985	0.966	0.986
400, rook, 0.4, 0.4, 0.8	0.648	0.977	0.994	0.998	0.981	0.991	1.000	0.998	0.998	0.999	0.999
400, rook, 0.4, 0.8, 0.2	0.524	0.485	0.537	0.545	0.501	0.540	0.517	0.556	0.380	0.455	0.501
400, rook, 0.4, 0.8, 0.4	0.933	0.960	0.976	0.979	0.950	0.976	0.974	0.979	0.936	0.963	0.976
400, rook, 0.4, 0.8, 0.6	0.913	0.999	1.000	1.000	0.995	1.000	1.000	1.000	0.999	1.000	1.000
400, rook, 0.4, 0.8, 0.8	0.872	0.999	0.999	1.000	0.999	0.998	1.000	0.998	1.000	0.997	1.000
400, rook, 0.8, 0.4, 0.2	0.654	0.645	0.644	0.647	0.650	0.630	0.640	0.643	0.650	0.639	0.673
400, rook, 0.8, 0.4, 0.4	0.997	0.996	0.996	0.997	0.995	0.993	0.997	0.996	0.997	0.996	0.997
400, rook, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	1.000	1.000	1.000	1.000	0.998	0.994	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.2	0.736	0.721	0.722	0.725	0.726	0.709	0.692	0.731	0.645	0.682	0.728
400, rook, 0.8, 0.8, 0.4	1.000	1.000	1.000	1.000	0.998	0.999	0.999	1.000	0.997	0.999	0.998
400, rook, 0.8, 0.8, 0.6	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.992	1.000	1.000	1.000	0.995	1.000	1.000	1.000	1.000	0.999	1.000

(1) S_{2SLS}, S_{GMM1}, S_{GMM2} and S_{QML} denote score-based OPG tests with, respectively, the 2SLS, GMM1, GMM2 and QML estimates of the SAR model;
(2) G_{GMM1} and G_{GMM2} denote gradient-based OPG tests with, respectively, the GMM1 and GMM2 estimates of the SAR model;
(3) D_{GMM1} and D_{GMM2} denote distance difference tests based on different moment vectors;
(4) W_{GMM1} and W_{GMM2} denote Wald tests based on the GMM estimation of the SARAR model with different moment vectors;
(5) W_{QML} denotes the Wald test based on the QML estimation of the SARAR model.

Table 7: Powers of tests for $\rho_0 = 0$ with normal heteroskedastic disturbances

n, W_n, R^2, λ_0, ρ_0	S_{2SLS}	S_{RGMM1}	S_{RGMM2}	S_{MQML}	G_{RGMM1}	G_{RGMM2}	D_{RGMM1}	D_{RGMM2}	W_{RGMM1}	W_{RGMM2}	W_{MQML}
144, queen, 0.4, 0.4, 0.2	0.123	0.064	0.076	0.090	0.094	0.073	0.080	0.080	0.125	0.098	0.200
144, queen, 0.4, 0.4, 0.4	0.298	0.222	0.248	0.277	0.228	0.244	0.288	0.274	0.278	0.221	0.415
144, queen, 0.4, 0.4, 0.6	0.483	0.411	0.452	0.538	0.394	0.431	0.578	0.562	0.514	0.449	0.659
144, queen, 0.4, 0.4, 0.8	0.493	0.475	0.617	0.726	0.464	0.583	0.808	0.795	0.716	0.620	0.722
144, queen, 0.4, 0.8, 0.2	0.134	0.088	0.129	0.152	0.101	0.142	0.173	0.182	0.155	0.133	0.243
144, queen, 0.4, 0.8, 0.4	0.337	0.321	0.434	0.480	0.320	0.429	0.473	0.497	0.330	0.258	0.542
144, queen, 0.4, 0.8, 0.6	0.527	0.622	0.760	0.799	0.523	0.743	0.802	0.842	0.601	0.496	0.762
144, queen, 0.4, 0.8, 0.8	0.797	0.867	0.934	0.962	0.717	0.905	0.961	0.954	0.829	0.685	0.901
144, queen, 0.8, 0.4, 0.2	0.308	0.155	0.152	0.169	0.135	0.147	0.154	0.152	0.196	0.191	0.227
144, queen, 0.8, 0.4, 0.4	0.741	0.530	0.520	0.550	0.481	0.504	0.551	0.562	0.595	0.574	0.610
144, queen, 0.8, 0.4, 0.6	0.953	0.844	0.846	0.873	0.790	0.812	0.877	0.882	0.906	0.875	0.885
144, queen, 0.8, 0.4, 0.8	0.992	0.958	0.948	0.977	0.836	0.857	0.982	0.980	0.991	0.977	0.974
144, queen, 0.8, 0.8, 0.2	0.328	0.150	0.161	0.183	0.136	0.160	0.160	0.191	0.192	0.169	0.218
144, queen, 0.8, 0.8, 0.4	0.763	0.581	0.589	0.623	0.510	0.581	0.584	0.628	0.605	0.598	0.664
144, queen, 0.8, 0.8, 0.6	0.944	0.865	0.871	0.895	0.783	0.844	0.879	0.908	0.881	0.884	0.906
144, queen, 0.8, 0.8, 0.8	0.934	0.977	0.977	0.992	0.862	0.942	0.983	0.983	0.982	0.983	0.977
144, rook, 0.4, 0.4, 0.2	0.121	0.137	0.147	0.166	0.154	0.154	0.146	0.155	0.157	0.112	0.261
144, rook, 0.4, 0.4, 0.4	0.343	0.357	0.377	0.424	0.394	0.378	0.466	0.462	0.418	0.352	0.545
144, rook, 0.4, 0.4, 0.6	0.499	0.538	0.636	0.711	0.590	0.620	0.811	0.778	0.750	0.632	0.789
144, rook, 0.4, 0.4, 0.8	0.500	0.562	0.721	0.844	0.607	0.667	0.921	0.895	0.878	0.787	0.760
144, rook, 0.4, 0.8, 0.2	0.160	0.155	0.220	0.229	0.191	0.218	0.253	0.265	0.201	0.183	0.265
144, rook, 0.4, 0.8, 0.4	0.456	0.496	0.644	0.676	0.534	0.639	0.680	0.684	0.506	0.451	0.629
144, rook, 0.4, 0.8, 0.6	0.673	0.805	0.917	0.933	0.750	0.899	0.933	0.954	0.850	0.786	0.873
144, rook, 0.4, 0.8, 0.8	0.846	0.920	0.939	0.992	0.825	0.933	0.993	0.968	0.976	0.879	0.925
144, rook, 0.8, 0.4, 0.2	0.329	0.223	0.228	0.245	0.229	0.231	0.226	0.248	0.274	0.263	0.306
144, rook, 0.8, 0.4, 0.4	0.852	0.750	0.754	0.788	0.757	0.728	0.789	0.788	0.831	0.797	0.809
144, rook, 0.8, 0.4, 0.6	0.973	0.944	0.943	0.960	0.888	0.872	0.967	0.966	0.980	0.965	0.970
144, rook, 0.8, 0.4, 0.8	0.928	0.963	0.934	0.995	0.835	0.772	0.998	0.995	0.999	0.996	0.989
144, rook, 0.8, 0.8, 0.2	0.389	0.269	0.278	0.302	0.272	0.282	0.262	0.305	0.303	0.286	0.356
144, rook, 0.8, 0.8, 0.4	0.859	0.791	0.810	0.822	0.746	0.790	0.763	0.835	0.800	0.791	0.849
144, rook, 0.8, 0.8, 0.6	0.944	0.960	0.963	0.983	0.886	0.932	0.962	0.979	0.972	0.965	0.974
144, rook, 0.8, 0.8, 0.8	0.877	0.983	0.980	0.999	0.852	0.930	0.998	0.993	1.000	0.997	0.992
400, queen, 0.4, 0.4, 0.2	0.256	0.194	0.202	0.210	0.189	0.188	0.224	0.220	0.203	0.168	0.264
400, queen, 0.4, 0.4, 0.4	0.658	0.583	0.597	0.607	0.562	0.573	0.639	0.636	0.566	0.512	0.628
400, queen, 0.4, 0.4, 0.6	0.858	0.826	0.845	0.882	0.811	0.823	0.913	0.910	0.877	0.843	0.843
400, queen, 0.4, 0.4, 0.8	0.770	0.895	0.942	0.967	0.876	0.925	0.982	0.969	0.970	0.961	0.834
400, queen, 0.4, 0.8, 0.2	0.246	0.249	0.285	0.297	0.244	0.287	0.284	0.328	0.216	0.230	0.279
400, queen, 0.4, 0.8, 0.4	0.676	0.747	0.823	0.829	0.724	0.823	0.817	0.857	0.737	0.764	0.773
400, queen, 0.4, 0.8, 0.6	0.842	0.950	0.991	0.993	0.954	0.990	0.988	0.989	0.974	0.974	0.925
400, queen, 0.4, 0.8, 0.8	0.898	0.992	0.998	1.000	0.987	0.996	1.000	0.996	0.999	0.987	0.916
400, queen, 0.8, 0.4, 0.2	0.553	0.405	0.402	0.411	0.381	0.393	0.410	0.421	0.427	0.416	0.435
400, queen, 0.8, 0.4, 0.4	0.972	0.926	0.928	0.934	0.908	0.921	0.931	0.933	0.931	0.923	0.931
400, queen, 0.8, 0.4, 0.6	1.000	0.999	0.998	0.998	0.994	0.994	0.999	0.999	0.999	0.998	0.997
400, queen, 0.8, 0.4, 0.8	1.000	1.000	0.999	1.000	0.991	0.995	1.000	1.000	1.000	1.000	0.999
400, queen, 0.8, 0.8, 0.2	0.562	0.425	0.432	0.442	0.396	0.423	0.411	0.465	0.397	0.422	0.450
400, queen, 0.8, 0.8, 0.4	0.978	0.939	0.943	0.949	0.935	0.937	0.939	0.951	0.936	0.934	0.942
400, queen, 0.8, 0.8, 0.6	1.000	1.000	1.000	1.000	0.997	0.996	1.000	1.000	0.999	1.000	0.999
400, queen, 0.8, 0.8, 0.8	0.980	0.999	0.999	1.000	0.990	0.993	1.000	1.000	1.000	1.000	0.998
400, rook, 0.4, 0.4, 0.2	0.275	0.327	0.337	0.354	0.347	0.339	0.352	0.361	0.290	0.251	0.373
400, rook, 0.4, 0.4, 0.4	0.661	0.800	0.810	0.823	0.796	0.790	0.851	0.846	0.790	0.746	0.796
400, rook, 0.4, 0.4, 0.6	0.724	0.954	0.966	0.976	0.943	0.950	0.989	0.984	0.986	0.961	0.943
400, rook, 0.4, 0.4, 0.8	0.575	0.894	0.973	0.998	0.932	0.958	0.999	0.993	0.998	0.997	0.840
400, rook, 0.4, 0.8, 0.2	0.392	0.465	0.507	0.527	0.451	0.509	0.497	0.552	0.374	0.456	0.466
400, rook, 0.4, 0.8, 0.4	0.779	0.934	0.972	0.975	0.933	0.970	0.958	0.972	0.931	0.951	0.917
400, rook, 0.4, 0.8, 0.6	0.818	0.982	1.000	1.000	0.991	0.999	0.999	0.998	0.998	0.999	0.962
400, rook, 0.4, 0.8, 0.8	0.841	0.999	0.999	1.000	0.991	0.997	1.000	1.000	1.000	0.996	0.947
400, rook, 0.8, 0.4, 0.2	0.681	0.625	0.620	0.631	0.628	0.611	0.630	0.633	0.640	0.620	0.648
400, rook, 0.8, 0.4, 0.4	0.998	0.990	0.990	0.991	0.988	0.985	0.998	0.993	0.996	0.993	0.992
400, rook, 0.8, 0.4, 0.6	0.999	0.999	0.998	1.000	0.992	0.994	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	1.000	0.999	0.998	1.000	0.974	0.955	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.2	0.737	0.663	0.666	0.679	0.691	0.666	0.671	0.694	0.643	0.670	0.688
400, rook, 0.8, 0.8, 0.4	0.995	0.994	0.994	0.992	0.990	0.993	0.995	0.995	0.992	0.992	0.994
400, rook, 0.8, 0.8, 0.6	0.997	1.000	1.000	1.000	0.998	0.998	1.000	1.000	1.000	1.000	0.999
400, rook, 0.8, 0.8, 0.8	0.939	0.999	0.998	1.000	0.990	0.992	1.000	0.999	1.000	1.000	0.998

(1) S_{2SLS}, S_{RGMM1}, S_{RGMM2} and S_{MQML} denote score-based OPG tests with, respectively, the 2SLS, RGMM1, RGMM2 and MQML estimates of the SAR model;

(2) G_{RGMM1} and G_{RGMM2} denote gradient-based OPG tests with, respectively, the RGMM1 and RGMM2 estimates of the SAR model;

(3) D_{RGMM1} and D_{RGMM2} denote distance difference tests based on different moment vectors;

(4) W_{RGMM1} and W_{RGMM2} denote Wald tests based on the robust GMM estimation of the SARAR model with different moment vectors;

(5) W_{MQML} denotes the Wald test based on the MQML estimation of the SARAR model.

Table 8: Powers of tests for $\rho_0 = 0$ with chi-squared heteroskedastic disturbances

n, W_n, R^2, λ_0, ρ_0	S_{2SLS}	S_{RGMM1}	S_{RGMM2}	S_{MQML}	G_{RGMM1}	G_{RGMM2}	D_{RGMM1}	D_{RGMM2}	W_{RGMM1}	W_{RGMM2}	W_{MQML}
144, queen, 0.4, 0.4, 0.2	0.151	0.077	0.087	0.106	0.128	0.099	0.113	0.101	0.151	0.105	0.220
144, queen, 0.4, 0.4, 0.4	0.371	0.236	0.279	0.336	0.266	0.285	0.335	0.328	0.357	0.301	0.465
144, queen, 0.4, 0.4, 0.6	0.519	0.396	0.459	0.550	0.435	0.462	0.633	0.595	0.585	0.486	0.677
144, queen, 0.4, 0.4, 0.8	0.526	0.470	0.598	0.749	0.509	0.561	0.827	0.787	0.783	0.666	0.733
144, queen, 0.4, 0.8, 0.2	0.137	0.100	0.169	0.198	0.127	0.171	0.189	0.178	0.195	0.144	0.273
144, queen, 0.4, 0.8, 0.4	0.376	0.289	0.467	0.523	0.350	0.464	0.528	0.508	0.436	0.324	0.544
144, queen, 0.4, 0.8, 0.6	0.571	0.578	0.787	0.826	0.593	0.764	0.840	0.830	0.688	0.541	0.795
144, queen, 0.4, 0.8, 0.8	0.813	0.868	0.940	0.978	0.716	0.894	0.975	0.945	0.896	0.767	0.912
144, queen, 0.8, 0.4, 0.2	0.341	0.188	0.179	0.217	0.179	0.176	0.208	0.207	0.258	0.245	0.265
144, queen, 0.8, 0.4, 0.4	0.759	0.560	0.566	0.601	0.541	0.548	0.593	0.602	0.651	0.618	0.647
144, queen, 0.8, 0.4, 0.6	0.946	0.857	0.856	0.877	0.831	0.842	0.909	0.891	0.930	0.888	0.874
144, queen, 0.8, 0.4, 0.8	0.978	0.965	0.961	0.982	0.872	0.879	0.982	0.983	0.989	0.985	0.977
144, queen, 0.8, 0.8, 0.2	0.329	0.155	0.169	0.194	0.181	0.182	0.187	0.190	0.246	0.187	0.252
144, queen, 0.8, 0.8, 0.4	0.752	0.567	0.583	0.616	0.540	0.573	0.593	0.617	0.624	0.579	0.647
144, queen, 0.8, 0.8, 0.6	0.944	0.878	0.892	0.908	0.834	0.858	0.898	0.917	0.906	0.887	0.903
144, queen, 0.8, 0.8, 0.8	0.941	0.968	0.980	0.996	0.874	0.881	0.992	0.985	0.994	0.989	0.979
144, rook, 0.4, 0.4, 0.2	0.167	0.151	0.161	0.184	0.210	0.168	0.203	0.201	0.210	0.172	0.306
144, rook, 0.4, 0.4, 0.4	0.396	0.423	0.453	0.515	0.478	0.447	0.556	0.537	0.522	0.440	0.612
144, rook, 0.4, 0.4, 0.6	0.523	0.562	0.658	0.742	0.645	0.596	0.825	0.801	0.785	0.686	0.786
144, rook, 0.4, 0.4, 0.8	0.540	0.515	0.720	0.873	0.627	0.687	0.927	0.901	0.908	0.836	0.777
144, rook, 0.4, 0.8, 0.2	0.190	0.160	0.231	0.257	0.225	0.238	0.280	0.269	0.251	0.200	0.305
144, rook, 0.4, 0.8, 0.4	0.486	0.404	0.682	0.737	0.547	0.670	0.751	0.739	0.624	0.548	0.686
144, rook, 0.4, 0.8, 0.6	0.700	0.740	0.921	0.946	0.755	0.896	0.959	0.950	0.899	0.820	0.889
144, rook, 0.4, 0.8, 0.8	0.858	0.884	0.949	0.993	0.750	0.918	0.986	0.964	0.980	0.892	0.914
144, rook, 0.8, 0.4, 0.2	0.393	0.299	0.299	0.333	0.338	0.318	0.313	0.325	0.390	0.369	0.410
144, rook, 0.8, 0.4, 0.4	0.821	0.765	0.767	0.804	0.779	0.755	0.803	0.797	0.850	0.807	0.818
144, rook, 0.8, 0.4, 0.6	0.938	0.938	0.936	0.950	0.892	0.881	0.967	0.960	0.983	0.959	0.953
144, rook, 0.8, 0.4, 0.8	0.931	0.958	0.931	0.994	0.856	0.767	0.994	0.994	0.998	0.995	0.978
144, rook, 0.8, 0.8, 0.2	0.407	0.286	0.300	0.339	0.317	0.307	0.295	0.317	0.357	0.319	0.410
144, rook, 0.8, 0.8, 0.4	0.841	0.794	0.806	0.822	0.783	0.800	0.792	0.830	0.820	0.792	0.840
144, rook, 0.8, 0.8, 0.6	0.919	0.963	0.966	0.985	0.898	0.920	0.971	0.983	0.981	0.975	0.969
144, rook, 0.8, 0.8, 0.8	0.869	0.969	0.972	1.000	0.797	0.902	0.995	0.990	0.999	0.994	0.987
400, queen, 0.4, 0.4, 0.2	0.268	0.235	0.235	0.252	0.249	0.237	0.259	0.250	0.236	0.206	0.298
400, queen, 0.4, 0.4, 0.4	0.665	0.586	0.599	0.623	0.638	0.607	0.689	0.674	0.587	0.537	0.639
400, queen, 0.4, 0.4, 0.6	0.861	0.844	0.869	0.897	0.858	0.854	0.937	0.924	0.883	0.852	0.856
400, queen, 0.4, 0.4, 0.8	0.770	0.853	0.927	0.963	0.870	0.894	0.978	0.971	0.972	0.960	0.812
400, queen, 0.4, 0.8, 0.2	0.294	0.255	0.334	0.362	0.287	0.328	0.351	0.365	0.283	0.298	0.325
400, queen, 0.4, 0.8, 0.4	0.683	0.680	0.818	0.835	0.735	0.813	0.837	0.857	0.728	0.733	0.759
400, queen, 0.4, 0.8, 0.6	0.850	0.894	0.988	0.993	0.938	0.979	0.989	0.991	0.978	0.983	0.938
400, queen, 0.4, 0.8, 0.8	0.900	0.994	1.000	1.000	0.967	0.994	1.000	0.984	1.000	0.984	0.936
400, queen, 0.8, 0.4, 0.2	0.528	0.423	0.422	0.427	0.422	0.416	0.428	0.425	0.445	0.421	0.453
400, queen, 0.8, 0.4, 0.4	0.971	0.936	0.935	0.938	0.933	0.935	0.954	0.947	0.956	0.942	0.931
400, queen, 0.8, 0.4, 0.6	0.999	0.996	0.996	0.997	0.993	0.990	0.998	0.997	0.999	0.996	0.993
400, queen, 0.8, 0.4, 0.8	1.000	1.000	0.999	1.000	0.994	0.986	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.2	0.544	0.432	0.443	0.457	0.418	0.436	0.429	0.450	0.420	0.433	0.480
400, queen, 0.8, 0.8, 0.4	0.983	0.954	0.952	0.955	0.938	0.949	0.953	0.969	0.955	0.950	0.954
400, queen, 0.8, 0.8, 0.6	0.996	0.997	1.000	1.000	0.991	0.994	1.000	1.000	1.000	1.000	0.999
400, queen, 0.8, 0.8, 0.8	0.978	0.998	1.000	1.000	0.984	0.993	1.000	0.999	1.000	1.000	0.991
400, rook, 0.4, 0.4, 0.2	0.306	0.349	0.350	0.372	0.397	0.362	0.387	0.378	0.322	0.298	0.400
400, rook, 0.4, 0.4, 0.4	0.670	0.767	0.781	0.801	0.820	0.787	0.850	0.843	0.790	0.746	0.764
400, rook, 0.4, 0.4, 0.6	0.713	0.925	0.949	0.971	0.935	0.925	0.981	0.976	0.973	0.953	0.930
400, rook, 0.4, 0.4, 0.8	0.586	0.891	0.967	0.995	0.943	0.938	1.000	0.990	0.999	0.993	0.863
400, rook, 0.4, 0.8, 0.2	0.421	0.456	0.523	0.548	0.503	0.533	0.544	0.570	0.423	0.475	0.516
400, rook, 0.4, 0.8, 0.4	0.749	0.873	0.970	0.975	0.925	0.965	0.965	0.970	0.930	0.958	0.923
400, rook, 0.4, 0.8, 0.6	0.795	0.949	0.999	1.000	0.971	0.991	1.000	0.998	0.999	0.997	0.957
400, rook, 0.4, 0.8, 0.8	0.841	0.993	0.997	1.000	0.976	0.991	1.000	0.997	1.000	0.994	0.932
400, rook, 0.8, 0.4, 0.2	0.693	0.643	0.642	0.661	0.702	0.644	0.675	0.672	0.690	0.666	0.678
400, rook, 0.8, 0.4, 0.4	0.989	0.993	0.993	0.991	0.986	0.979	0.995	0.994	0.994	0.991	0.988
400, rook, 0.8, 0.4, 0.6	0.994	0.999	0.998	0.999	0.994	0.978	1.000	1.000	0.999	0.999	0.998
400, rook, 0.8, 0.4, 0.8	0.984	1.000	0.998	1.000	0.986	0.941	1.000	1.000	1.000	1.000	0.999
400, rook, 0.8, 0.8, 0.2	0.754	0.705	0.710	0.717	0.727	0.717	0.716	0.731	0.705	0.698	0.723
400, rook, 0.8, 0.8, 0.4	0.990	0.994	0.996	0.996	0.984	0.992	0.994	0.996	0.997	0.995	0.995
400, rook, 0.8, 0.8, 0.6	0.989	0.999	0.999	1.000	0.994	0.991	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.921	0.995	0.999	1.000	0.958	0.988	1.000	1.000	1.000	1.000	0.999

(1) S_{2SLS}, S_{RGMM1}, S_{RGMM2} and S_{MQML} denote score-based OPG tests with, respectively, the 2SLS, RGMM1, RGMM2 and MQML estimates of the SAR model;

(2) G_{RGMM1} and G_{RGMM2} denote gradient-based OPG tests with, respectively, the RGMM1 and RGMM2 estimates of the SAR model;

(3) D_{RGMM1} and D_{RGMM2} denote distance difference tests based on different moment vectors;

(4) W_{RGMM1} and W_{RGMM2} denote Wald tests based on the robust GMM estimation of the SARAR model with different moment vectors;

(5) W_{MQML} denotes the Wald test based on the MQML estimation of the SARAR model.

Table 9: Powers of tests for $\lambda_0 = 0$ with normal homoskedastic disturbances

n, W_n, R^2, ρ_0, λ_0	S_{LS}	S_{GMM1}	S_{GMM2}	S_{QML}	G_{GMM1}	G_{GMM2}	D_{GMM1}	D_{GMM2}	W_{GMM1}	W_{GMM2}	W_{QML}
144, queen, 0.4, 0.4, 0.2	0.126	0.092	0.126	0.122	0.085	0.096	0.093	0.094	0.258	0.237	0.211
144, queen, 0.4, 0.4, 0.4	0.271	0.218	0.273	0.269	0.225	0.242	0.263	0.257	0.442	0.427	0.413
144, queen, 0.4, 0.4, 0.6	0.416	0.361	0.470	0.455	0.364	0.408	0.511	0.509	0.629	0.592	0.606
144, queen, 0.4, 0.4, 0.8	0.490	0.623	0.692	0.685	0.582	0.632	0.808	0.821	0.803	0.758	0.808
144, queen, 0.4, 0.8, 0.2	0.156	0.145	0.196	0.192	0.094	0.112	0.179	0.211	0.197	0.183	0.194
144, queen, 0.4, 0.8, 0.4	0.341	0.389	0.493	0.489	0.237	0.256	0.529	0.570	0.373	0.329	0.389
144, queen, 0.4, 0.8, 0.6	0.588	0.761	0.745	0.823	0.412	0.408	0.890	0.917	0.612	0.574	0.718
144, queen, 0.4, 0.8, 0.8	0.750	0.956	0.675	0.989	0.682	0.475	0.982	0.979	0.902	0.875	0.973
144, queen, 0.8, 0.4, 0.2	0.328	0.289	0.324	0.319	0.283	0.318	0.294	0.299	0.502	0.469	0.431
144, queen, 0.8, 0.4, 0.4	0.808	0.755	0.806	0.800	0.722	0.786	0.751	0.774	0.898	0.883	0.871
144, queen, 0.8, 0.4, 0.6	0.946	0.931	0.956	0.955	0.913	0.947	0.967	0.961	0.991	0.989	0.988
144, queen, 0.8, 0.4, 0.8	0.896	0.992	0.996	0.996	0.954	0.994	0.994	0.998	0.999	0.999	1.000
144, queen, 0.8, 0.8, 0.2	0.259	0.275	0.301	0.300	0.254	0.305	0.280	0.310	0.397	0.367	0.358
144, queen, 0.8, 0.8, 0.4	0.713	0.777	0.806	0.808	0.700	0.799	0.763	0.807	0.841	0.831	0.816
144, queen, 0.8, 0.8, 0.6	0.895	0.980	0.980	0.986	0.938	0.977	0.962	0.983	0.992	0.994	0.988
144, queen, 0.8, 0.8, 0.8	0.797	0.998	0.897	1.000	0.992	0.951	0.996	0.993	1.000	1.000	1.000
144, rook, 0.4, 0.4, 0.2	0.209	0.191	0.210	0.210	0.150	0.159	0.170	0.158	0.260	0.237	0.202
144, rook, 0.4, 0.4, 0.4	0.508	0.483	0.533	0.527	0.451	0.476	0.515	0.544	0.568	0.537	0.537
144, rook, 0.4, 0.4, 0.6	0.655	0.707	0.786	0.776	0.693	0.732	0.820	0.818	0.834	0.801	0.828
144, rook, 0.4, 0.4, 0.8	0.452	0.881	0.930	0.936	0.868	0.910	0.958	0.962	0.938	0.935	0.954
144, rook, 0.4, 0.8, 0.2	0.242	0.287	0.343	0.344	0.188	0.191	0.326	0.369	0.238	0.231	0.281
144, rook, 0.4, 0.8, 0.4	0.507	0.715	0.765	0.780	0.465	0.504	0.801	0.825	0.601	0.611	0.715
144, rook, 0.4, 0.8, 0.6	0.636	0.953	0.880	0.980	0.753	0.700	0.988	0.992	0.935	0.921	0.973
144, rook, 0.4, 0.8, 0.8	0.536	0.997	0.649	0.998	0.917	0.622	0.995	0.991	0.996	0.993	0.999
144, rook, 0.8, 0.4, 0.2	0.623	0.610	0.620	0.620	0.604	0.614	0.584	0.601	0.726	0.709	0.663
144, rook, 0.8, 0.4, 0.4	0.971	0.964	0.972	0.971	0.948	0.969	0.963	0.974	0.990	0.989	0.983
144, rook, 0.8, 0.4, 0.6	0.993	0.999	0.999	0.999	0.993	0.998	0.996	0.999	0.999	0.999	0.999
144, rook, 0.8, 0.4, 0.8	0.813	1.000	1.000	1.000	0.998	0.998	1.000	1.000	1.000	1.000	1.000
144, rook, 0.8, 0.8, 0.2	0.537	0.627	0.657	0.658	0.567	0.659	0.591	0.622	0.662	0.668	0.677
144, rook, 0.8, 0.8, 0.4	0.887	0.986	0.992	0.992	0.971	0.987	0.974	0.986	0.986	0.987	0.991
144, rook, 0.8, 0.8, 0.6	0.804	1.000	0.993	1.000	0.999	0.996	0.999	1.000	0.999	1.000	1.000
144, rook, 0.8, 0.8, 0.8	0.400	1.000	0.881	1.000	0.997	0.960	1.000	1.000	1.000	1.000	1.000
400, queen, 0.4, 0.4, 0.2	0.195	0.184	0.185	0.184	0.173	0.174	0.211	0.197	0.298	0.288	0.267
400, queen, 0.4, 0.4, 0.4	0.568	0.542	0.573	0.569	0.503	0.509	0.581	0.585	0.653	0.634	0.639
400, queen, 0.4, 0.4, 0.6	0.810	0.797	0.829	0.822	0.815	0.835	0.880	0.883	0.905	0.893	0.904
400, queen, 0.4, 0.4, 0.8	0.885	0.924	0.979	0.973	0.954	0.974	0.972	0.982	0.972	0.978	0.984
400, queen, 0.4, 0.8, 0.2	0.290	0.285	0.335	0.322	0.160	0.181	0.277	0.342	0.216	0.220	0.259
400, queen, 0.4, 0.8, 0.4	0.758	0.764	0.852	0.843	0.516	0.572	0.758	0.856	0.675	0.721	0.794
400, queen, 0.4, 0.8, 0.6	0.970	0.969	0.997	0.997	0.821	0.870	0.987	0.997	0.968	0.980	0.996
400, queen, 0.4, 0.8, 0.8	0.977	1.000	0.995	1.000	0.981	0.960	0.999	0.999	1.000	1.000	1.000
400, queen, 0.8, 0.4, 0.2	0.692	0.681	0.690	0.687	0.671	0.684	0.667	0.672	0.762	0.766	0.730
400, queen, 0.8, 0.4, 0.4	0.994	0.993	0.995	0.995	0.991	0.995	0.995	0.994	0.998	0.997	0.997
400, queen, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.4, 0.8	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.2	0.586	0.595	0.614	0.610	0.546	0.613	0.554	0.612	0.602	0.609	0.626
400, queen, 0.8, 0.8, 0.4	0.989	0.997	0.997	0.997	0.990	0.997	0.990	0.996	0.991	0.992	0.995
400, queen, 0.8, 0.8, 0.6	0.999	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.8	0.986	1.000	0.988	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.4, 0.4, 0.2	0.403	0.398	0.401	0.401	0.347	0.340	0.396	0.396	0.402	0.395	0.375
400, rook, 0.4, 0.4, 0.4	0.852	0.856	0.861	0.860	0.833	0.841	0.877	0.891	0.887	0.872	0.878
400, rook, 0.4, 0.4, 0.6	0.961	0.983	0.983	0.983	0.987	0.993	0.990	0.993	0.993	0.995	0.996
400, rook, 0.4, 0.4, 0.8	0.795	0.999	1.000	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000
400, rook, 0.4, 0.8, 0.2	0.539	0.592	0.613	0.610	0.401	0.420	0.563	0.640	0.470	0.536	0.567
400, rook, 0.4, 0.8, 0.4	0.952	0.978	0.989	0.989	0.878	0.897	0.979	0.991	0.964	0.984	0.986
400, rook, 0.4, 0.8, 0.6	0.964	1.000	1.000	1.000	0.997	0.996	0.999	1.000	0.999	1.000	1.000
400, rook, 0.4, 0.8, 0.8	0.787	1.000	0.992	1.000	1.000	0.997	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.2	0.954	0.955	0.955	0.955	0.953	0.951	0.956	0.959	0.968	0.967	0.960
400, rook, 0.8, 0.4, 0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	0.982	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.2	0.946	0.958	0.961	0.961	0.947	0.965	0.946	0.959	0.946	0.956	0.957
400, rook, 0.8, 0.8, 0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.6	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.554	1.000	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

(1) S_{2SLS}, S_{GMM1}, S_{GMM2} and S_{QML} denote score-based OPG tests with, respectively, the 2SLS, GMM1, GMM2 and QML estimates of the SAR model;
(2) G_{GMM1} and G_{GMM2} denote gradient-based OPG tests with, respectively, the GMM1 and GMM2 estimates of the SAR model;
(3) D_{GMM1} and D_{GMM2} denote distance difference tests based on different moment vectors;
(4) W_{GMM1} and W_{GMM2} denote Wald tests based on the GMM estimation of the SARAR model with different moment vectors;
(5) W_{QML} denotes the Wald test based on the QML estimation of the SARAR model.

Table 10: Powers of tests for $\lambda_0 = 0$ with chi-squared homoskedastic disturbances

n, W_n, R^2, ρ_0, λ_0	S_{LS}	S_{GMM1}	S_{GMM2}	S_{QML}	G_{GMM1}	G_{GMM2}	D_{GMM1}	D_{GMM2}	W_{GMM1}	W_{GMM2}	W_{QML}
144, queen, 0.4, 0.4, 0.2	0.127	0.049	0.130	0.126	0.092	0.086	0.089	0.087	0.271	0.233	0.210
144, queen, 0.4, 0.4, 0.4	0.312	0.116	0.310	0.299	0.223	0.270	0.276	0.271	0.499	0.462	0.446
144, queen, 0.4, 0.4, 0.6	0.458	0.263	0.516	0.492	0.418	0.459	0.533	0.533	0.651	0.634	0.630
144, queen, 0.4, 0.4, 0.8	0.476	0.559	0.713	0.703	0.618	0.654	0.848	0.844	0.836	0.798	0.832
144, queen, 0.4, 0.8, 0.2	0.161	0.098	0.198	0.187	0.102	0.111	0.179	0.191	0.208	0.186	0.217
144, queen, 0.4, 0.8, 0.4	0.376	0.361	0.514	0.506	0.238	0.255	0.546	0.567	0.384	0.346	0.412
144, queen, 0.4, 0.8, 0.6	0.607	0.755	0.785	0.844	0.475	0.470	0.896	0.913	0.668	0.623	0.781
144, queen, 0.4, 0.8, 0.8	0.758	0.970	0.641	0.997	0.697	0.464	0.987	0.981	0.905	0.867	0.973
144, queen, 0.8, 0.4, 0.2	0.345	0.288	0.327	0.324	0.320	0.332	0.289	0.295	0.511	0.484	0.425
144, queen, 0.8, 0.4, 0.4	0.808	0.719	0.805	0.805	0.748	0.804	0.774	0.789	0.940	0.913	0.883
144, queen, 0.8, 0.4, 0.6	0.946	0.922	0.949	0.946	0.912	0.946	0.971	0.967	0.991	0.989	0.985
144, queen, 0.8, 0.4, 0.8	0.880	0.981	0.991	0.990	0.960	0.990	0.995	0.995	0.999	0.997	0.999
144, queen, 0.8, 0.8, 0.2	0.284	0.261	0.322	0.319	0.262	0.316	0.277	0.287	0.438	0.400	0.354
144, queen, 0.8, 0.8, 0.4	0.734	0.743	0.802	0.796	0.741	0.800	0.778	0.806	0.879	0.848	0.826
144, queen, 0.8, 0.8, 0.6	0.882	0.965	0.972	0.981	0.940	0.972	0.964	0.983	0.989	0.991	0.985
144, queen, 0.8, 0.8, 0.8	0.779	0.999	0.904	0.999	0.986	0.955	0.999	0.996	1.000	0.997	0.999
144, rook, 0.4, 0.4, 0.2	0.213	0.153	0.216	0.218	0.171	0.172	0.156	0.168	0.275	0.241	0.221
144, rook, 0.4, 0.4, 0.4	0.512	0.396	0.546	0.543	0.478	0.477	0.544	0.521	0.639	0.586	0.583
144, rook, 0.4, 0.4, 0.6	0.645	0.634	0.788	0.785	0.730	0.750	0.848	0.820	0.849	0.811	0.845
144, rook, 0.4, 0.4, 0.8	0.424	0.871	0.905	0.927	0.888	0.904	0.963	0.963	0.946	0.932	0.947
144, rook, 0.4, 0.8, 0.2	0.199	0.211	0.326	0.326	0.181	0.191	0.336	0.339	0.226	0.221	0.278
144, rook, 0.4, 0.8, 0.4	0.542	0.657	0.778	0.795	0.499	0.500	0.787	0.827	0.609	0.635	0.725
144, rook, 0.4, 0.8, 0.6	0.666	0.952	0.858	0.986	0.775	0.703	0.989	0.992	0.918	0.940	0.975
144, rook, 0.4, 0.8, 0.8	0.525	0.997	0.606	0.999	0.903	0.529	0.996	0.991	0.989	0.986	0.998
144, rook, 0.8, 0.4, 0.2	0.642	0.617	0.642	0.645	0.637	0.649	0.622	0.626	0.777	0.737	0.685
144, rook, 0.8, 0.4, 0.4	0.974	0.969	0.980	0.980	0.971	0.981	0.978	0.980	0.992	0.991	0.987
144, rook, 0.8, 0.4, 0.6	0.994	0.998	0.999	0.999	0.986	0.999	0.998	0.999	1.000	0.999	0.999
144, rook, 0.8, 0.4, 0.8	0.850	1.000	0.999	1.000	0.989	0.999	1.000	1.000	1.000	1.000	1.000
144, rook, 0.8, 0.8, 0.2	0.566	0.622	0.678	0.679	0.633	0.672	0.634	0.649	0.747	0.718	0.685
144, rook, 0.8, 0.8, 0.4	0.890	0.972	0.980	0.982	0.965	0.977	0.972	0.983	0.991	0.987	0.981
144, rook, 0.8, 0.8, 0.6	0.798	1.000	0.990	1.000	0.998	0.993	0.999	1.000	1.000	1.000	1.000
144, rook, 0.8, 0.8, 0.8	0.384	1.000	0.895	1.000	0.996	0.951	0.999	1.000	1.000	1.000	1.000
400, queen, 0.4, 0.4, 0.2	0.224	0.189	0.217	0.216	0.188	0.200	0.195	0.208	0.317	0.313	0.279
400, queen, 0.4, 0.4, 0.4	0.558	0.452	0.560	0.556	0.493	0.517	0.575	0.577	0.651	0.632	0.612
400, queen, 0.4, 0.4, 0.6	0.830	0.727	0.846	0.843	0.820	0.856	0.896	0.896	0.920	0.899	0.907
400, queen, 0.4, 0.4, 0.8	0.870	0.890	0.965	0.962	0.947	0.966	0.967	0.974	0.959	0.968	0.968
400, queen, 0.4, 0.8, 0.2	0.279	0.230	0.314	0.306	0.179	0.182	0.280	0.319	0.205	0.213	0.257
400, queen, 0.4, 0.8, 0.4	0.763	0.712	0.831	0.825	0.530	0.577	0.776	0.842	0.641	0.713	0.782
400, queen, 0.4, 0.8, 0.6	0.977	0.956	0.994	0.995	0.820	0.851	0.986	0.994	0.951	0.978	0.992
400, queen, 0.4, 0.8, 0.8	0.979	1.000	0.993	1.000	0.977	0.958	0.999	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.4, 0.2	0.691	0.659	0.681	0.680	0.675	0.690	0.673	0.674	0.788	0.762	0.737
400, queen, 0.8, 0.4, 0.4	0.985	0.970	0.982	0.982	0.981	0.984	0.988	0.989	0.997	0.995	0.993
400, queen, 0.8, 0.4, 0.6	1.000	0.994	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.4, 0.8	0.998	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.2	0.587	0.579	0.615	0.612	0.588	0.623	0.593	0.611	0.622	0.615	0.633
400, queen, 0.8, 0.8, 0.4	0.985	0.987	0.990	0.990	0.981	0.988	0.987	0.992	0.985	0.987	0.993
400, queen, 0.8, 0.8, 0.6	0.997	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.8	0.972	1.000	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.4, 0.4, 0.2	0.397	0.377	0.393	0.392	0.351	0.362	0.384	0.379	0.414	0.402	0.369
400, rook, 0.4, 0.4, 0.4	0.861	0.849	0.875	0.873	0.838	0.858	0.892	0.890	0.914	0.891	0.896
400, rook, 0.4, 0.4, 0.6	0.966	0.977	0.987	0.987	0.987	0.986	0.989	0.985	0.990	0.987	0.990
400, rook, 0.4, 0.4, 0.8	0.781	0.999	1.000	1.000	0.999	1.000	1.000	0.999	1.000	1.000	1.000
400, rook, 0.4, 0.8, 0.2	0.526	0.543	0.609	0.607	0.392	0.423	0.570	0.623	0.441	0.506	0.564
400, rook, 0.4, 0.8, 0.4	0.942	0.964	0.991	0.990	0.878	0.897	0.976	0.989	0.935	0.981	0.986
400, rook, 0.4, 0.8, 0.6	0.952	0.999	1.000	1.000	0.992	0.990	1.000	1.000	0.999	1.000	1.000
400, rook, 0.4, 0.8, 0.8	0.768	1.000	0.984	1.000	1.000	0.993	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.2	0.973	0.972	0.974	0.974	0.972	0.976	0.975	0.975	0.986	0.983	0.977
400, rook, 0.8, 0.4, 0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	0.983	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.2	0.933	0.950	0.954	0.954	0.947	0.958	0.949	0.954	0.941	0.946	0.953
400, rook, 0.8, 0.8, 0.4	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.6	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.549	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

(1) S_{2SLS}, S_{GMM1}, S_{GMM2} and S_{QML} denote score-based OPG tests with, respectively, the 2SLS, GMM1, GMM2 and QML estimates of the SAR model;
(2) G_{GMM1} and G_{GMM2} denote gradient-based OPG tests with, respectively, the GMM1 and GMM2 estimates of the SAR model;
(3) D_{GMM1} and D_{GMM2} denote distance difference tests based on different moment vectors;
(4) W_{GMM1} and W_{GMM2} denote Wald tests based on the GMM estimation of the SARAR model with different moment vectors;
(5) W_{QML} denotes the Wald test based on the QML estimation of the SARAR model.

Table 11: Powers of tests for $\lambda_0 = 0$ with normal heteroskedastic disturbances

n, W_n, R^2, ρ_0, λ_0	S_{LS}	S_{RGMM1}	S_{RGMM2}	S_{MQML}	G_{RGMM1}	G_{RGMM2}	D_{RGMM1}	D_{RGMM2}	W_{RGMM1}	W_{RGMM2}	W_{MQML}
144, queen, 0.4, 0.4, 0.2	0.134	0.089	0.136	0.133	0.098	0.126	0.112	0.104	0.341	0.274	0.297
144, queen, 0.4, 0.4, 0.4	0.310	0.207	0.326	0.315	0.246	0.320	0.295	0.264	0.541	0.470	0.568
144, queen, 0.4, 0.4, 0.6	0.460	0.332	0.519	0.501	0.421	0.542	0.569	0.538	0.714	0.638	0.695
144, queen, 0.4, 0.4, 0.8	0.488	0.546	0.729	0.727	0.596	0.761	0.824	0.837	0.777	0.709	0.732
144, queen, 0.4, 0.8, 0.2	0.153	0.119	0.214	0.200	0.139	0.189	0.209	0.222	0.341	0.296	0.340
144, queen, 0.4, 0.8, 0.4	0.351	0.377	0.512	0.513	0.357	0.462	0.551	0.616	0.510	0.446	0.616
144, queen, 0.4, 0.8, 0.6	0.582	0.736	0.736	0.838	0.591	0.698	0.891	0.908	0.727	0.682	0.830
144, queen, 0.4, 0.8, 0.8	0.708	0.959	0.672	0.986	0.881	0.629	0.990	0.988	0.901	0.864	0.892
144, queen, 0.8, 0.4, 0.2	0.396	0.325	0.382	0.373	0.362	0.385	0.369	0.355	0.618	0.566	0.510
144, queen, 0.8, 0.4, 0.4	0.800	0.734	0.798	0.790	0.793	0.812	0.824	0.815	0.946	0.928	0.890
144, queen, 0.8, 0.4, 0.6	0.910	0.873	0.939	0.937	0.933	0.946	0.955	0.946	0.993	0.985	0.962
144, queen, 0.8, 0.4, 0.8	0.858	0.968	0.985	0.985	0.951	0.990	0.991	0.990	0.998	0.999	0.951
144, queen, 0.8, 0.8, 0.2	0.292	0.307	0.346	0.339	0.288	0.315	0.331	0.340	0.533	0.483	0.416
144, queen, 0.8, 0.8, 0.4	0.715	0.737	0.788	0.791	0.709	0.744	0.784	0.802	0.895	0.875	0.816
144, queen, 0.8, 0.8, 0.6	0.867	0.953	0.968	0.976	0.926	0.941	0.954	0.971	0.993	0.986	0.960
144, queen, 0.8, 0.8, 0.8	0.782	0.998	0.898	0.999	0.986	0.930	0.997	0.997	1.000	1.000	0.974
144, rook, 0.4, 0.4, 0.2	0.209	0.158	0.201	0.201	0.184	0.201	0.200	0.187	0.356	0.319	0.349
144, rook, 0.4, 0.4, 0.4	0.484	0.399	0.515	0.513	0.455	0.516	0.518	0.499	0.673	0.624	0.658
144, rook, 0.4, 0.4, 0.6	0.598	0.573	0.763	0.753	0.660	0.800	0.790	0.789	0.871	0.796	0.802
144, rook, 0.4, 0.4, 0.8	0.379	0.792	0.898	0.919	0.784	0.919	0.936	0.949	0.925	0.907	0.804
144, rook, 0.4, 0.8, 0.2	0.176	0.217	0.315	0.315	0.224	0.292	0.326	0.360	0.439	0.406	0.358
144, rook, 0.4, 0.8, 0.4	0.457	0.617	0.737	0.781	0.598	0.716	0.776	0.806	0.743	0.741	0.736
144, rook, 0.4, 0.8, 0.6	0.529	0.903	0.821	0.966	0.843	0.833	0.964	0.977	0.929	0.919	0.893
144, rook, 0.4, 0.8, 0.8	0.435	0.992	0.589	1.000	0.943	0.572	0.992	0.989	0.982	0.976	0.930
144, rook, 0.8, 0.4, 0.2	0.648	0.612	0.644	0.640	0.643	0.649	0.648	0.638	0.802	0.770	0.708
144, rook, 0.8, 0.4, 0.4	0.942	0.928	0.948	0.947	0.948	0.953	0.971	0.960	0.994	0.986	0.964
144, rook, 0.8, 0.4, 0.6	0.977	0.988	0.993	0.993	0.988	0.994	0.996	0.997	1.000	0.999	0.989
144, rook, 0.8, 0.4, 0.8	0.792	0.995	1.000	1.000	0.992	0.999	1.000	1.000	1.000	1.000	0.980
144, rook, 0.8, 0.8, 0.2	0.498	0.564	0.606	0.606	0.590	0.578	0.611	0.602	0.734	0.697	0.643
144, rook, 0.8, 0.8, 0.4	0.822	0.961	0.972	0.973	0.954	0.959	0.966	0.962	0.989	0.983	0.966
144, rook, 0.8, 0.8, 0.6	0.751	0.991	0.990	0.998	0.985	0.991	0.996	0.999	1.000	0.999	0.994
144, rook, 0.8, 0.8, 0.8	0.355	0.997	0.888	1.000	0.976	0.898	0.999	0.999	1.000	0.999	0.986
400, queen, 0.4, 0.4, 0.2	0.218	0.192	0.220	0.219	0.193	0.209	0.189	0.199	0.369	0.332	0.357
400, queen, 0.4, 0.4, 0.4	0.567	0.491	0.573	0.571	0.553	0.584	0.605	0.598	0.725	0.690	0.709
400, queen, 0.4, 0.4, 0.6	0.802	0.659	0.835	0.828	0.792	0.868	0.863	0.876	0.904	0.877	0.850
400, queen, 0.4, 0.4, 0.8	0.870	0.855	0.963	0.962	0.919	0.974	0.956	0.964	0.981	0.980	0.798
400, queen, 0.4, 0.8, 0.2	0.264	0.251	0.334	0.318	0.230	0.307	0.289	0.347	0.341	0.355	0.310
400, queen, 0.4, 0.8, 0.4	0.749	0.671	0.827	0.822	0.645	0.775	0.760	0.826	0.744	0.787	0.730
400, queen, 0.4, 0.8, 0.6	0.958	0.954	0.989	0.992	0.954	0.990	0.978	0.986	0.986	0.990	0.900
400, queen, 0.4, 0.8, 0.8	0.967	0.999	0.979	1.000	0.998	0.981	1.000	1.000	1.000	0.998	0.895
400, queen, 0.8, 0.4, 0.2	0.669	0.638	0.662	0.660	0.660	0.665	0.672	0.675	0.792	0.786	0.729
400, queen, 0.8, 0.4, 0.4	0.979	0.976	0.980	0.979	0.983	0.980	0.990	0.983	0.996	0.994	0.985
400, queen, 0.8, 0.4, 0.6	0.998	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.4, 0.8	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.988
400, queen, 0.8, 0.8, 0.2	0.587	0.599	0.627	0.622	0.564	0.609	0.623	0.639	0.669	0.675	0.635
400, queen, 0.8, 0.8, 0.4	0.970	0.978	0.981	0.980	0.970	0.976	0.981	0.988	0.988	0.985	0.978
400, queen, 0.8, 0.8, 0.6	0.998	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, queen, 0.8, 0.8, 0.8	0.962	1.000	0.991	1.000	0.998	0.998	0.999	1.000	1.000	1.000	0.992
400, rook, 0.4, 0.4, 0.2	0.398	0.361	0.391	0.389	0.382	0.381	0.410	0.377	0.504	0.452	0.469
400, rook, 0.4, 0.4, 0.4	0.821	0.811	0.847	0.844	0.857	0.857	0.878	0.877	0.919	0.892	0.843
400, rook, 0.4, 0.4, 0.6	0.937	0.938	0.981	0.980	0.964	0.980	0.984	0.981	0.990	0.989	0.936
400, rook, 0.4, 0.4, 0.8	0.682	0.979	0.999	0.999	0.987	0.999	0.994	0.995	0.999	0.998	0.772
400, rook, 0.4, 0.8, 0.2	0.441	0.501	0.564	0.562	0.467	0.535	0.533	0.576	0.515	0.579	0.482
400, rook, 0.4, 0.8, 0.4	0.881	0.936	0.979	0.980	0.940	0.971	0.953	0.960	0.958	0.981	0.871
400, rook, 0.4, 0.8, 0.6	0.886	1.000	0.999	1.000	0.998	1.000	0.999	0.999	1.000	1.000	0.944
400, rook, 0.4, 0.8, 0.8	0.701	1.000	0.975	1.000	0.998	0.976	1.000	1.000	1.000	1.000	0.931
400, rook, 0.8, 0.4, 0.2	0.926	0.922	0.921	0.920	0.939	0.927	0.946	0.935	0.966	0.955	0.935
400, rook, 0.8, 0.4, 0.4	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	0.999
400, rook, 0.8, 0.4, 0.6	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	0.976	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
400, rook, 0.8, 0.8, 0.2	0.898	0.927	0.931	0.930	0.924	0.918	0.931	0.943	0.937	0.946	0.928
400, rook, 0.8, 0.8, 0.4	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
400, rook, 0.8, 0.8, 0.6	0.973	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.503	1.000	0.996	1.000	1.000	0.999	1.000	1.000	1.000	1.000	0.999

(1) S_{2SLS}, S_{RGMM1}, S_{RGMM2} and S_{MQML} denote score-based OPG tests with, respectively, the 2SLS, RGMM1, RGMM2 and MQML estimates of the SAR model;
(2) G_{RGMM1} and G_{RGMM2} denote gradient-based OPG tests with, respectively, the RGMM1 and RGMM2 estimates of the SAR model;
(3) D_{RGMM1} and D_{RGMM2} denote distance difference tests based on different moment vectors;
(4) W_{RGMM1} and W_{RGMM2} denote Wald tests based on the robust GMM estimation of the SARAR model with different moment vectors;
(5) W_{MQML} denotes the Wald test based on the MQML estimation of the SARAR model.

Table 12: Powers of tests for $\lambda_0 = 0$ with chi-squared heteroskedastic disturbances

n, W_n, R^2, ρ_0, λ_0	S_{LS}	S_{RGMM1}	S_{RGMM2}	S_{MQML}	G_{RGMM1}	G_{RGMM2}	D_{RGMM1}	D_{RGMM2}	W_{RGMM1}	W_{RGMM2}	W_{MQML}
144, queen, 0.4, 0.4, 0.2	0.125	0.063	0.122	0.117	0.099	0.116	0.085	0.089	0.331	0.279	0.298
144, queen, 0.4, 0.4, 0.4	0.278	0.135	0.289	0.278	0.237	0.282	0.233	0.230	0.540	0.496	0.554
144, queen, 0.4, 0.4, 0.6	0.441	0.261	0.535	0.505	0.410	0.560	0.507	0.502	0.746	0.674	0.734
144, queen, 0.4, 0.4, 0.8	0.475	0.546	0.768	0.744	0.633	0.803	0.855	0.852	0.854	0.782	0.745
144, queen, 0.4, 0.8, 0.2	0.096	0.066	0.153	0.140	0.082	0.113	0.118	0.130	0.276	0.250	0.332
144, queen, 0.4, 0.8, 0.4	0.295	0.296	0.490	0.468	0.323	0.406	0.502	0.522	0.491	0.447	0.624
144, queen, 0.4, 0.8, 0.6	0.551	0.683	0.737	0.827	0.641	0.701	0.896	0.899	0.788	0.694	0.824
144, queen, 0.4, 0.8, 0.8	0.673	0.963	0.664	0.987	0.866	0.583	0.991	0.994	0.959	0.915	0.914
144, queen, 0.8, 0.4, 0.2	0.390	0.321	0.376	0.366	0.359	0.365	0.330	0.343	0.628	0.588	0.522
144, queen, 0.8, 0.4, 0.4	0.831	0.763	0.834	0.828	0.840	0.841	0.832	0.843	0.969	0.949	0.916
144, queen, 0.8, 0.4, 0.6	0.937	0.909	0.968	0.965	0.970	0.978	0.972	0.974	0.997	0.995	0.988
144, queen, 0.8, 0.4, 0.8	0.856	0.977	0.991	0.991	0.989	0.999	0.995	0.995	1.000	0.999	0.955
144, queen, 0.8, 0.8, 0.2	0.234	0.228	0.286	0.280	0.250	0.250	0.261	0.273	0.498	0.439	0.375
144, queen, 0.8, 0.8, 0.4	0.690	0.715	0.810	0.800	0.734	0.742	0.758	0.777	0.940	0.906	0.831
144, queen, 0.8, 0.8, 0.6	0.864	0.972	0.982	0.983	0.950	0.962	0.961	0.982	0.997	0.999	0.977
144, queen, 0.8, 0.8, 0.8	0.745	1.000	0.923	1.000	0.976	0.929	0.994	0.996	0.999	0.999	0.981
144, rook, 0.4, 0.4, 0.2	0.200	0.156	0.202	0.198	0.201	0.193	0.150	0.163	0.346	0.296	0.368
144, rook, 0.4, 0.4, 0.4	0.466	0.377	0.506	0.499	0.503	0.527	0.494	0.481	0.708	0.625	0.671
144, rook, 0.4, 0.4, 0.6	0.582	0.593	0.787	0.782	0.745	0.819	0.835	0.807	0.912	0.842	0.868
144, rook, 0.4, 0.4, 0.8	0.325	0.801	0.903	0.914	0.809	0.924	0.942	0.945	0.973	0.959	0.816
144, rook, 0.4, 0.8, 0.2	0.137	0.170	0.271	0.269	0.215	0.232	0.259	0.279	0.337	0.335	0.362
144, rook, 0.4, 0.8, 0.4	0.402	0.579	0.721	0.753	0.579	0.650	0.753	0.754	0.754	0.744	0.760
144, rook, 0.4, 0.8, 0.6	0.460	0.892	0.824	0.957	0.830	0.809	0.965	0.974	0.962	0.957	0.923
144, rook, 0.4, 0.8, 0.8	0.423	0.976	0.593	0.982	0.897	0.547	0.998	0.994	0.998	0.987	0.947
144, rook, 0.8, 0.4, 0.2	0.663	0.602	0.657	0.654	0.687	0.663	0.653	0.630	0.847	0.792	0.729
144, rook, 0.8, 0.4, 0.4	0.974	0.969	0.981	0.981	0.981	0.985	0.976	0.985	0.998	0.998	0.986
144, rook, 0.8, 0.4, 0.6	0.980	0.990	0.997	0.997	0.994	0.997	0.997	0.997	1.000	1.000	0.997
144, rook, 0.8, 0.4, 0.8	0.803	0.997	0.998	0.998	0.985	1.000	0.998	0.999	1.000	1.000	0.979
144, rook, 0.8, 0.8, 0.2	0.489	0.579	0.632	0.634	0.617	0.613	0.612	0.594	0.787	0.741	0.687
144, rook, 0.8, 0.8, 0.4	0.834	0.968	0.982	0.982	0.975	0.982	0.976	0.977	0.998	0.999	0.983
144, rook, 0.8, 0.8, 0.6	0.700	0.995	0.994	0.996	0.990	0.996	0.994	0.996	1.000	1.000	0.996
144, rook, 0.8, 0.8, 0.8	0.312	0.997	0.923	0.998	0.982	0.942	0.998	0.999	1.000	1.000	0.995
400, queen, 0.4, 0.4, 0.2	0.204	0.163	0.207	0.203	0.190	0.200	0.167	0.159	0.331	0.325	0.353
400, queen, 0.4, 0.4, 0.4	0.548	0.446	0.557	0.546	0.527	0.582	0.535	0.539	0.736	0.695	0.703
400, queen, 0.4, 0.4, 0.6	0.811	0.565	0.848	0.839	0.803	0.883	0.872	0.874	0.930	0.917	0.857
400, queen, 0.4, 0.4, 0.8	0.848	0.813	0.965	0.963	0.926	0.980	0.967	0.967	0.981	0.983	0.788
400, queen, 0.4, 0.8, 0.2	0.244	0.180	0.306	0.296	0.192	0.257	0.223	0.271	0.267	0.281	0.318
400, queen, 0.4, 0.8, 0.4	0.716	0.616	0.830	0.823	0.602	0.762	0.745	0.791	0.724	0.778	0.744
400, queen, 0.4, 0.8, 0.6	0.928	0.945	0.989	0.992	0.944	0.975	0.984	0.983	0.967	0.989	0.929
400, queen, 0.4, 0.8, 0.8	0.945	1.000	0.984	1.000	0.997	0.982	0.999	0.998	1.000	0.999	0.900
400, queen, 0.8, 0.4, 0.2	0.710	0.677	0.706	0.704	0.705	0.707	0.702	0.705	0.830	0.810	0.771
400, queen, 0.8, 0.4, 0.4	0.992	0.980	0.991	0.991	0.992	0.997	0.992	0.992	1.000	0.999	0.995
400, queen, 0.8, 0.4, 0.6	0.997	0.990	0.998	0.998	1.000	0.999	0.999	1.000	1.000	1.000	0.999
400, queen, 0.8, 0.4, 0.8	0.985	1.000	1.000	1.000	0.999	0.999	1.000	1.000	1.000	1.000	0.983
400, queen, 0.8, 0.8, 0.2	0.612	0.581	0.650	0.647	0.571	0.595	0.613	0.628	0.704	0.701	0.673
400, queen, 0.8, 0.8, 0.4	0.980	0.988	0.991	0.992	0.982	0.985	0.988	0.988	0.992	0.990	0.992
400, queen, 0.8, 0.8, 0.6	0.992	1.000	0.999	1.000	1.000	0.999	1.000	1.000	1.000	1.000	0.998
400, queen, 0.8, 0.8, 0.8	0.952	1.000	0.992	1.000	0.999	0.998	1.000	1.000	1.000	1.000	0.994
400, rook, 0.4, 0.4, 0.2	0.384	0.374	0.392	0.391	0.404	0.379	0.365	0.363	0.490	0.483	0.483
400, rook, 0.4, 0.4, 0.4	0.848	0.825	0.867	0.865	0.867	0.890	0.880	0.882	0.933	0.917	0.880
400, rook, 0.4, 0.4, 0.6	0.911	0.909	0.972	0.970	0.970	0.984	0.988	0.981	0.997	0.994	0.946
400, rook, 0.4, 0.4, 0.8	0.609	0.981	0.996	0.995	0.980	0.994	0.998	0.988	0.996	0.999	0.802
400, rook, 0.4, 0.8, 0.2	0.439	0.469	0.567	0.564	0.490	0.502	0.533	0.571	0.517	0.596	0.524
400, rook, 0.4, 0.8, 0.4	0.854	0.922	0.984	0.986	0.934	0.958	0.960	0.957	0.953	0.984	0.887
400, rook, 0.4, 0.8, 0.6	0.824	0.999	0.997	0.998	0.995	0.996	0.998	0.998	0.998	1.000	0.940
400, rook, 0.4, 0.8, 0.8	0.644	1.000	0.977	1.000	0.991	0.980	1.000	0.999	1.000	1.000	0.929
400, rook, 0.8, 0.4, 0.2	0.972	0.966	0.973	0.973	0.971	0.975	0.976	0.970	0.985	0.981	0.976
400, rook, 0.8, 0.4, 0.4	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	0.999
400, rook, 0.8, 0.4, 0.6	0.997	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.4, 0.8	0.936	1.000	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.995
400, rook, 0.8, 0.8, 0.2	0.921	0.945	0.952	0.952	0.947	0.945	0.946	0.952	0.966	0.965	0.952
400, rook, 0.8, 0.8, 0.4	0.978	1.000	1.000	1.000	1.000	0.999	1.000	0.999	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.6	0.921	1.000	1.000	1.000	0.999	0.999	1.000	1.000	1.000	1.000	1.000
400, rook, 0.8, 0.8, 0.8	0.405	1.000	0.996	1.000	1.000	0.999	1.000	1.000	1.000	1.000	0.998

(1) S_{2SLS}, S_{RGMM1}, S_{RGMM2} and S_{MQML} denote score-based OPG tests with, respectively, the 2SLS, RGMM1, RGMM2 and MQML estimates of the SAR model;

(2) G_{RGMM1} and G_{RGMM2} denote gradient-based OPG tests with, respectively, the RGMM1 and RGMM2 estimates of the SAR model;

(3) D_{RGMM1} and D_{RGMM2} denote distance difference tests based on different moment vectors;

(4) W_{RGMM1} and W_{RGMM2} denote Wald tests based on the robust GMM estimation of the SARAR model with different moment vectors;

(5) W_{MQML} denotes the Wald test based on the MQML estimation of the SARAR model.